**Math 601 – Spring 2026 – Harry Tamvakis**
**PROBLEM SET 10 – Due April 30, 2026**

**A1)** Suppose that $G$ is the infinite group $\left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \;\middle|\; n \in \mathbb{Z} \right\}$ (note that $G$ is isomorphic to the additive group $(\mathbb{Z}, +)$.) Let $V$ be the $\mathbb{C}G$-module $\mathbb{C}^2$, with the natural multiplication by elements of $G$. Show that $V$ is not a direct sum of irreducible representations of $G$ (this shows that Maschke's theorem fails for infinite groups.)

**A2)** Suppose that

$$G = D_4 = \langle x, y \mid x^4 = y^2 = 1,\ yx = x^{-1}y \rangle .$$

Check that there is a matrix representation $\rho : G \to \mathrm{GL}_2(\mathbb{C})$ such that

$$\rho(x) = \begin{pmatrix} -7 & 10 \\ -5 & 7 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} -5 & 6 \\ -4 & 5 \end{pmatrix}.$$

Find all $2 \times 2$ matrices $A$ such that $A\rho(g) = \rho(g)A$ for all $g \in G$. Hence determine whether or not $\rho$ is irreducible.

Do the same for the representation $\sigma$ of $G$, where

$$\sigma(x) = \begin{pmatrix} 5 & -6 \\ 4 & -5 \end{pmatrix}, \quad \sigma(y) = \begin{pmatrix} -5 & 6 \\ -4 & 5 \end{pmatrix}.$$

**A3)** For each $n \geq 2$, let $D_n$ be the dihedral group of order $2n$. Show that $D_n$ has exactly two one-dimensional representations, if $n$ is odd, and exactly four one-dimensional representations, if $n$ is even.

**A4)** Consider the standard two dimensional representation of the dihedral group $D_n$ as symmetries of the regular $n$-gon. For which values of $n \geq 3$ is it irreducible as a complex representation?

**A5)** Let $G$ be a finite group. If $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ is an irreducible matrix representation and $A$ is an $n \times n$ matrix commuting with $\rho(g)$ for all $g \in G$, prove that $A$ is a scalar matrix. [Hint: Show that the eigenspaces of $A$ are $G$-stable.] Deduce that if $\rho$ is a faithful, irreducible, complex representation then the center $Z$ of $G$ is cyclic and $\rho(g)$ is a scalar matrix for all $g \in Z$.

**A6)** Recall from homework set #7 that a complex number $\lambda$ is an *algebraic integer* if it is the root of a monic polynomial with integer coefficients.

(a) Prove that $\lambda$ is an algebraic integer if and only if $\lambda$ is an eigenvalue of some matrix $A$, all of whose entries are integers.

(b) If $\lambda$, $\mu$ are eigenvalues of matrices $A \in M_r(\mathbb{Z})$ and $B \in M_s(\mathbb{Z})$ respectively, prove that $\lambda\mu$ is an eigenvalue of the matrix $A \otimes B$ and $\lambda + \mu$ is an eigenvalue of the matrix $A \otimes I_s + I_r \otimes B$. Deduce that the set of algebraic integers is a commutative ring.

(c) Let $G$ be a finite group and $\chi$ a character of $G$. If $g \in G$, show that $\chi(g)$ is an algebraic integer. Deduce that if $\chi(g)$ is a rational number, then $\chi(g)$ is an integer.

**A7)** Let $G$ be a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$. Prove that if $\sum_{g \in G} \mathrm{Tr}(g) = 0$, then $\sum_{g \in G} g = 0$.

**B1)** Suppose $x_1, \ldots, x_n$ are $n$ variables. For each $k$ with $1 \leq k \leq n$ let $e_k(x_1, \ldots, x_n)$ be the $k$-th elementary symmetric polynomial and define the $k$-th *power sum* by

$$p_k(x_1, \ldots, x_n) := x_1^k + \cdots + x_n^k.$$

(a) Prove *Newton's identities:*

$$p_k - e_1 p_{k-1} + e_2 p_{k-2} - \cdots + (-1)^{k-1} p_1 e_{k-1} + (-1)^k k e_k = 0$$

for $1 \leq k \leq n$.

(b) For $1 \leq k \leq n$, prove the determinantal formulas

$$p_k = \begin{vmatrix} e_1 & 2e_2 & 3e_3 & \cdots & & ke_k \\ 1 & e_1 & e_2 & \cdots & & e_{k-1} \\ 0 & 1 & e_1 & \cdots & & e_{k-2} \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & e_1 \end{vmatrix}$$

and

$$k!\, e_k = \begin{vmatrix} p_1 & p_2 & p_3 & \cdots & & p_k \\ 1 & p_1 & p_2 & \cdots & & p_{k-1} \\ 0 & 2 & p_1 & \cdots & & p_{k-2} \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 & k-1 & p_1 \end{vmatrix}.$$

(c) Deduce that if $\lambda_1, \ldots, \lambda_n$ are $n$ complex numbers such that $p_k(\lambda_1, \ldots, \lambda_n)$ is known for all $k$ with $1 \leq k \leq n$, then $\lambda_1, \ldots, \lambda_n$ are uniquely determined.

(d) Deduce further that the character $\chi : G \to \mathbb{C}$ of a representation $V$ of a finite group $G$ determines the eigenvalues of the action $g : V \to V$ of any element $g \in G$ on $V$.

**B2)** Let $V$ be an irreducible representation of the finite group $G$. Show that, up to scalars, there is a *unique* $G$-invariant hermitian inner product on $V$, i.e. such that $\langle gx, gy \rangle = \langle x, y \rangle$ for all $g \in G$ and $x, y \in V$.

**B3)** Let $G$ be a finite group and let $H$ be a normal subgroup of $G$ with $|G : H| = n$. Let $V$ be an $n$-dimensional complex vector space with a basis $b_{g_1 H}, \ldots, b_{g_n H}$ indexed by the cosets of $H$. Consider the representation $\rho : G \to \mathrm{GL}(V)$, where $\rho(g) : V \to V$ is defined by $\rho(g)(b_{g_i H}) := b_{gg_i H}$. Show that $\rho$ is irreducible if and only if $G = H$.

## C problem

**C1)** Suppose that $F$ is a proper subfield of $\mathbb{C}$ such that $\mathbb{C} \supset F$ is a finite extension. The goal of this problem is to prove that $[\mathbb{C} : F] = 2$.

(a) If $[\mathbb{C} : F] = p$ is prime, show that $\mathbb{C} \supset F$ is Galois with cyclic Galois group, and $F$ contains all $p$-th roots of unity.

(b) If $[\mathbb{C} : F] = p$ is prime and $\sigma$ is a generator of $G(\mathbb{C}/F)$, show that there exist an $\alpha \in \mathbb{C} \setminus F$, a $p$-th root $\beta$ of $\alpha$, and a primitive $p^2$-th root of unity $\gamma$ such that $\alpha^p \in F$ and $\sigma(\beta) = \gamma\beta$.

(c) Show that if $p$ is an odd prime, then no such $F$ exists.

(d) Show that $[\mathbb{C} : F] = 2$, and in fact $\mathbb{C} = F(i)$.

Remark: One can show more generally the following surprising result: If $K$ is any algebraically closed field which is a proper finite extension of a subfield $F$, then $K$ must have characteristic zero, so the above argument applies and $K = F(i)$ for some $i \in K$ such that $i^2 + 1 = 0$.