Math 606 – Fall 2026 – Harry Tamvakis
PROBLEM SET 10 – Due November 19, 2026

In the following, X is a compact Riemann surface of genus $g \geq 1$. Let P be a point of X and $1 < a_1 < a_2 < \cdots < a_g = 2g$ be the first g Weierstrass 'non-gaps'. That is, for each j, there exists a meromorphic function on X, holomorphic on $X \smallsetminus P$, with a pole of order a_j at P.

1) (a) Show that for each integer j with $1 \leq j \leq g-1$, we have

$$a_j + a_{g-j} \geq 2g.$$

(b) If $a_1 = 2$, show that $a_j = 2j$ and $a_j + a_{g-j} = 2g$ for $1 \leq j \leq g-1$.

(c) If $a_1 > 2$, show that for some j with $1 \leq j \leq g-1$, we have

$$a_j + a_{g-j} > 2g.$$

Deduce that

$$\sum_{j=1}^{g-1} a_j \geq g(g-1)$$

with equality if and only if $a_1 = 2$.

2) Prove that $n \geq 1$ is a Weierstrass 'gap' at P if and only if there exists a holomorphic 1-form with a zero of order $n-1$ at P. Deduce that there exists a holomorphic 1-form that does not vanish at P.

3) Let $P_1, \ldots, P_n$ be distinct points on X.

(a) Show that there exists a meromorphic function f on X with a pole at P_1 and zeroes at $P_2, \ldots, P_n$.

(b) If $z_1, \ldots, z_n$ are complex numbers, show that there exists a meromorphic function f on X such that $f(P_i) = z_i$ for $1 \leq i \leq n$.

4) Let $\mathcal{M}$ denote the sheaf of meromorphic functions on X and let $\mathcal{N}$ be the sheaf of meromorphic 1-forms which have residue 0 at every point of X. Prove that the sheaf sequence

$$0 \to \mathbb{C} \to \mathcal{M} \xrightarrow{d} \mathcal{N} \to 0$$

is exact. Deduce from this that $\mathrm{H}^1(X, \mathbb{C}) \cong \mathcal{N}(X)/d\mathcal{M}(X)$.

5) Suppose that X has genus 2 and that ω_1 and ω_2 form a basis of $\mathrm{H}^0(X, \Omega)$. Define $f \in \mathcal{M}(X)$ by $\omega_1 = f\omega_2$. Prove that $f : X \to \mathbb{P}^1$ is a 2-sheeted branched covering map.