Math 660 – Spring 2024 – Harry Tamvakis
PROBLEM SET 10 – Due April 25, 2024

1) Let $D = D(0,1)$ be the unit disk in $\mathbb{C}$ and $f \in \mathcal{O}(D)$. Suppose that $|\operatorname{Im} f(z)| < \pi/2$ for all $z \in D$, and that $f(0) = 0$. Prove that $|f'(0)| \leq 2$.

[Hint: Consider the function $g(z) = \dfrac{e^{f(z)} - 1}{e^{f(z)} + 1}$.]

2) Let Ω be a convex open set in $\mathbb{C}$ and $f : D(0,1) \to \Omega$ be a biholomorphic map. For each real number r with $0 < r < 1$, prove that $f(D(0,r))$ is convex.

[Hint: Assume that $f(0) = 0$, and for all $a, b \in D(0,1)$ with $|a| \leq |b| < 1$ and $b \neq 0$, consider $f^{-1}\left((1-t) \cdot f(ab^{-1}z) + t \cdot f(z)\right)$.]

3) Exercise 310 in the textbook.

4) Exercise 313 in the textbook.

5) Exercise 315 in the textbook.

Extra Credit Problem.

EC) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that for each fixed $x \in \mathbb{R}$, $y \mapsto f(x,y)$ is a polynomial in y, and for each fixed $y \in \mathbb{R}$, $x \mapsto f(x,y)$ is a polynomial in x. Prove that $f \in \mathbb{R}[x,y]$, i.e., that $f(x,y)$ is a polynomial in two variables.

Remark. If $\mathbb{R}$ is replaced by $\mathbb{C}$, the result follows immediately from Hartogs' theorem on separate analyticity, which was mentioned in the lectures without proof. Find a more elementary proof that works over $\mathbb{R}$ or $\mathbb{C}$.