Math 600 – Fall 2025 – Harry Tamvakis

PROBLEM SET 11 – Due December 4, 2025

A1) Let F be a field, f be an irreducible polynomial in $F[x]$, and t be a variable. Show that f is irreducible in $F(t)[x]$.

A2) Let V be the algebraic set in $\mathbb{A}^3(\mathbb{C})$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that V is a union of three irreducible components. Describe them and find their prime ideals.

A3) Let $I = (x^2 - y^3, y^2 - z^3) \subset \mathbb{C}[x, y, z]$. Define a map $\alpha : \mathbb{C}[x, y, z] \to \mathbb{C}[t]$ by $\alpha(x) = t^9$, $\alpha(y) = t^6$, $\alpha(z) = t^4$.

(a) Show that every element of $\mathbb{C}[x, y, z]/I$ is the residue of an element $a + xb + yc + xyd$, for some $a, b, c, d \in \mathbb{C}[z]$.

(b) If $f := a + xb + yc + xyd$, $a, b, c, d \in \mathbb{C}[z]$, and $\alpha(f) = 0$, compare like powers of t to conclude that $f = 0$.

(c) Show that $\mathrm{Ker}(\alpha) = I$. Deduce that I is prime and $V(I)$ is irreducible.

A4) (a) Let F be an infinite field and $f \in F[x_1, \ldots, x_n]$. Suppose that $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in F$. Prove that $f = 0$.

(b) Let f be a non-constant polynomial in $\mathbb{C}[x_1, \ldots, x_n]$. Show that $\mathbb{A}^n \smallsetminus V(f)$ is infinite if $n \geq 1$, and $V(f)$ is infinite if $n \geq 2$. Conclude that the complement of any algebraic set other than $\mathbb{A}^n$ is infinite.

A5) Suppose we choose the field $F = \mathbb{R}$ and work in the real affine plane $\mathbb{A}^2(\mathbb{R})$.

(a) Show that every algebraic subset of $\mathbb{A}^2(\mathbb{R})$ is equal to $V(f)$ for some $f \in \mathbb{R}[x, y]$.

(b) Show that both the weak and strong form of Hilbert's Nullstellensatz are false if we work with $\mathbb{R}$ instead of $\mathbb{C}$. Find a prime ideal I in $\mathbb{R}[x, y]$ so that $V(I)$ is reducible.

These results indicate why algebraic geometers prefer to work with an algebraically closed field like $\mathbb{C}$ (a field F is *algebraically closed* if every non-constant polynomial $f \in F[x]$ has a root in F).

B1) (a) Let R be a P.I.D. and let $\mathcal{I}$ be a set of representatives for its irreducible elements (up to associates). Let K be the quotient field of R, and let $a \in K$. Show that for every $p \in \mathcal{I}$ there exists an element $a_p \in R$ and an integer $k(p) \geq 0$, such that $k(p) = 0$ for all but finitely many p in $\mathcal{I}$,

a_p and $p^{k(p)}$ are relatively prime, and

$$a = \sum_{p \in \mathcal{I}} \frac{a_p}{p^{k(p)}}.$$

Furthermore, if we have another such expression $a = \sum_{p \in \mathcal{I}} \frac{b_p}{p^{\ell(p)}}$ then $k(p) = \ell(p)$ for all p, and $a_p = b_p \pmod{p^{k(p)}}$ for all p in $\mathcal{I}$.

(b) Let F be a field, $R := F[x]$, and let $\mathcal{I}$ denote the set of monic irreducible polynomials in $F[x]$. Show that any rational function $f \in F(x)$ has a unique expression

$$f(x) = \sum_{p \in \mathcal{I}} \frac{f_p(x)}{p(x)^{k(p)}} + g(x) \tag{1}$$

where f_p and g are polynomials, $f_p = 0$ if $k(p) = 0$, f_p is relatively prime to p if $k(p) > 0$, and $\deg f_p < \deg p^{k(p)}$ if $k(p) > 0$.

(c) One can further decompose the terms $f_p/p^{k(p)}$ in (1) by expanding f_p according to powers of p. In fact, show that if $f, g \in F[x]$ and $\deg g \geq 1$, then there exist unique polynomials $f_0, \ldots, f_d \in F[x]$ such that $\deg f_i < \deg g$ and

$$f = f_0 + f_1 g + \cdots f_d g^d. \tag{2}$$

The right hand side of equation (2) is called the *g-adic expansion of f*.

(d) Suppose that $f, g \in \mathbb{R}[x]$ and $\deg f < \deg g$. Show that one can write the fraction $\frac{f(x)}{g(x)}$ in $\mathbb{R}(x)$ as a sum of partial fractions of one of the forms $\frac{a}{(x-r)^m}$ or $\frac{bx+c}{(x^2+sx+t)^n}$ where $x^2 + sx + t$ is irreducible.

B2) Let R be a commutative ring.

(a) Prove that an element $x \in R$ belongs to every prime ideal of R if and only if $x^m = 0$ for some $m \geq 1$.

(b) If I is an ideal of R, show that the intersection of all prime ideals P which contain I is equal to the radical of I.

Note that if $R = \mathbb{C}[x_1, \ldots, x_n]$ one can replace the word 'prime' above by 'maximal' (compare with problem B3 of Problem Set 10).

B3) (a) Given a finite set of maximal ideals $m_1, \ldots, m_d$ of $\mathbb{C}[x, y]$, is there a non-zero prime ideal contained in each of them? Justify you answer.

(b) Answer the same question for the ring $\mathbb{C}[x_1, \ldots, x_n]$ where $n \geq 3$.