Math 601 – Spring 2026 – Harry Tamvakis
PROBLEM SET 11 – Due May 7, 2026

A1) Let V be the permutation representation associated to the action of the finite group G on a finite set X, and χ be the character of V. Let χ_1 denote the trivial character of G.

(a) Show that for $g \in G$, $\chi(g)$ is equal to the number of elements of X that are fixed by g.

(b) Show that $\langle \chi, \chi_1 \rangle$ is equal to the number of G-orbits in X.

A2) (a) Suppose G is a finite group and

$$G' = [G, G] = \langle xyx^{-1}y^{-1} \mid x, y \in G \rangle$$

is the commutator subgroup of G. Prove that the number of representations of degree 1 of G is equal to the index $|G : G'|$.

(b) Determine all degree 1 representations of the symmetric group S_n.

A3) Suppose G is a finite group such that for every normal subgroup N of G, the quotient group G/N is abelian. Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation of G with $\dim(V) > 1$. Prove that ρ is injective.

A4) Let $V \subset \mathbb{C}[x_1, x_2, x_3]$ be the 6-dimensional vector space of homogeneous polynomials in x_1, x_2, x_3 of degree 2 over $\mathbb{C}$. View V as a representation of S_3, with S_3 acting by permuting the variables. Determine the character χ_V of the representation V and express χ_V as a sum of irreducible characters.

A5) Let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be an irreducible representation of a finite group G with character χ, and let $\pi : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}(V)$ be any complex representation of $\mathrm{GL}_n(\mathbb{C})$.

(a) Determine the character of the representation $\pi \circ \rho$ of G when π is left multiplication of $\mathrm{GL}_n(\mathbb{C})$ on $V := M_n(\mathbb{C})$. Decompose $\pi \circ \rho$ as a direct sum of irreducible representations in this case.

(b) Determine the character of the representation $\pi \circ \rho$ when π is the operation of conjugation on $M_n(\mathbb{C})$.

B1) If $\rho : G \to \mathrm{GL}(V)$ is a representation of a finite group G, and $\phi : G \to \mathbb{C}$ is any function, define the *Fourier transform* $\widehat{\phi}(\rho)$ in $\mathrm{End}(V)$ by the formula

$$\widehat{\phi}(\rho) = \sum_{g \in G} \phi(g)\rho(g).$$

(a) Show that if ϕ, ψ are any two functions $G \to \mathbb{C}$, then

$$\widehat{\phi * \psi}(\rho) = \widehat{\phi}(\rho)\widehat{\psi}(\rho).$$

(b) Suppose that $\rho_i : G \to \mathrm{GL}(V_i)$ for $1 \leq i \leq k$ are the irreducible representations of G. Prove the *Fourier inversion formula*

$$\phi(g) = \frac{1}{|G|}\sum_{i=1}^{k}(\dim V_i) \cdot \mathrm{Tr}(\rho_i(g^{-1})\widehat{\phi}(\rho_i)).$$

Deduce that an element $g \in G$ is determined if we know the values $\rho_i(g)$, that is, if we know the matrices representing g in all the irreducible representations of G. [Hint: View $\mathbb{C}G$ as a direct sum of irreducible representations of G, and identify the right hand side with the trace of an operator acting on $\mathbb{C}G$.]

(c) Prove the *Plancherel formula* for functions ϕ and ψ on G:

$$\sum_{g\in G}\phi(g)\psi(g^{-1}) = \frac{1}{|G|}\sum_{i=1}^{k}(\dim V_i) \cdot \mathrm{Tr}(\widehat{\phi}(\rho_i)\widehat{\psi}(\rho_i)).$$

[Hint: First take $\phi \leftrightarrow e_g \in \mathbb{C}G$ to be the characteristic function of an element $g \in G$ and apply (b).]

To see how this problem is related to classical Fourier analysis, consider the case when $G = S^1 = \{z = e^{i\theta} \mid \theta \in \mathbb{R}\}$ is the unit circle in the complex plane. A representation of the compact group S^1 is a *continuous* group homomorphism $\rho : S^1 \to \mathrm{GL}_n(\mathbb{C})$. Since S^1 is an abelian group, all its irreducible representations are one dimensional, so they are given by continuous homomorphisms $S^1 \to \mathbb{C}^*$. One can easily show that all such maps are of the form $\rho_n(z) = z^n$ for $n \in \mathbb{Z}$. In other words, to each integer n there corresponds a unique irreducible representation of S^1, and vice-versa.

If $f : S^1 \to \mathbb{C}$ is a 'nice' function (say continuous or at least such that $|f|^2$ is integrable – this latter condition defines the space $L^2(S^1)$), the Fourier transforms of f as defined above give a sequence of complex numbers (called the 'Fourier coefficients' of f), namely

$$a_n = \widehat{f}(n) = \int_{S^1} f(z)z^n\,dz = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{in\theta}\,d\theta.$$

(in the second integral, we interpret f as a 2π-periodic function on the real θ-line.) The Fourier inversion formula says that f can be recovered from its Fourier coefficients:

$$f(\theta) = \sum_{n=-\infty}^{+\infty} a_n e^{-in\theta}.$$

The convergence of the infinite series must be interpreted in the right way, that is, in the 'L^2-sense'. In this context the decomposition of the regular representation of S^1 into a direct sum of irreducible representations is realized as the decomposition of the (infinite dimensional) *Hilbert space* $L^2(S^1)$ given by the orthonormal basis $\{\rho_n \mid n \in \mathbb{Z}\}$. This subject is one of the beautiful meeting grounds of algebra and analysis.

B2) Let G be a finite group and suppose that V is a faithful representation of G, i.e., $\rho : G \to \mathrm{GL}(V)$ is injective. Our goal is to prove that any irreducible representation of G is contained in some tensor power $V^{\otimes n}$ of V (this is a theorem of Burnside and Molien).

(a) If χ is the character of V and ψ is any irreducible character, let $a_n = \langle \chi^n, \psi \rangle$ and consider the *generating function* $f(t) := \sum a_n t^n$. Show that

$$f(t) = \sum_{n=0}^{\infty} a_n t^n = \frac{1}{|G|} \sum_{C} \frac{|C|\overline{\psi(C)}}{1 - \chi(C)t}.$$

In the sum the index C runs over all conjugacy classes in G.

(b) Show that $\chi(C) = \dim V$ if and only if $C = [1]$.

(c) Deduce that $f(t)$ is a *non-constant* rational function, and finish the proof of the Burnside-Molien theorem.

B3) (a) Let G be a finite group and $g \in G$. Prove that g is conjugate to g^{-1} if and only if $\chi(g)$ is real for every irreducible character χ of G. We say that the conjugacy class of g is *self-inverse* in this case.

(b) Prove that the number of irreducible characters of G that take only real values (the so-called *real characters*) is equal to the number of self-inverse conjugacy classes of G (the so-called *real classes*).

C problems

C1) Let G be a finite group and χ be the character of an irreducible representation of G. Prove that if $|\chi(g)| = 1$ for some $g \in G$ then $\chi(g)$ is a root of unity. [Hint: Show that for each $n \in \mathbb{N}$, there are only finitely many monic polynomials of degree n in $\mathbb{Z}[x]$ all of whose roots have absolute value one. Then use the basic facts about the Galois group $G(\mathbb{Q}(\zeta_n)/\mathbb{Q})$, where $\zeta_n := e^{2\pi i/n}$, which were cited in class.]

C2) Let G be a finite group and χ be the character of an irreducible representation of G dimension greater than one. Prove that there exists some conjugacy class C such that $\chi(g) = 0$ for all $g \in C$. [Much easier: prove this under the additional assumption that $\chi(g)$ is a rational number for all $g \in G$.]