Math 660 – Spring 2024 – Harry Tamvakis
PROBLEM SET 11 – Due May 2, 2024

In the following problems, $\mathrm{GL}_2(\mathbb{Z})$ denotes the group of 2×2 invertible matrices with integer entries and $\mathrm{SL}_2(\mathbb{Z})$ denotes the subgroup of matrices A in $\mathrm{GL}_2(\mathbb{Z})$ with $\det(A) = 1$.

1) Let $\Lambda = \mathbb{Z}\,\omega_1 + \mathbb{Z}\,\omega_2$ and $\Lambda' = \mathbb{Z}\,\omega_1' + \mathbb{Z}\,\omega_2'$ be two lattices in $\mathbb{C}$. Show that $\Lambda = \Lambda'$ if and only if there exists a matrix $A \in \mathrm{GL}_2(\mathbb{Z})$ such that

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = A \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

2) Let Λ, Λ' be two lattices in $\mathbb{C}$ and $X = \mathbb{C}/\Lambda$, $X' = \mathbb{C}/\Lambda'$ the corresponding complex tori. Prove that any holomorphic map

$$f : X \to X'$$

is induced by a linear map $g : \mathbb{C} \to \mathbb{C}$ of the form $g(z) = \alpha z + \beta$, where $\alpha \in \mathbb{C}$ is such that $\alpha\Lambda \subset \Lambda'$. Show that the map f is biholomorphic if and only if $\alpha\Lambda = \Lambda'$.

3) (a) Prove that every torus $X = \mathbb{C}/\Lambda$ is biholomorphic to a torus of the form $X(\tau) := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, where $\tau \in \mathbb{C}$ satisfies $\mathrm{Im}(\tau) > 0$.

(b) Assume that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $\mathrm{Im}(\tau) > 0$, and let $\tau' := \dfrac{a\tau + b}{c\tau + d}$. Show that the tori $X(\tau)$ and $X(\tau')$ are biholomorphic.

4) Suppose that $p_1, \ldots, p_n$ are points on the compact Riemann surface X and $X' = X \smallsetminus \{p_1, \ldots, p_n\}$. Consider a non-constant holomorphic function $f : X' \to \mathbb{C}$. Show that the image of f comes arbitrarily close to every complex number c.

5) Determine the branch points (or ramification points) of the map $f : \mathbb{C} \to \mathbb{P}^1$ with

$$f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right).$$