

**Math 600 – Fall 2025 – Harry Tamvakis**  
**PROBLEM SET 12 – Due December 11, 2025**

**A1)** Let  $A$  be a commutative ring and  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  be an exact sequence of  $A$ -modules. Prove that if  $M'$  and  $M''$  are finitely generated, then  $M$  is finitely generated.

**A2)** Let  $F$  be a field and  $A := F[x_1, x_2, \dots]$  be the ring of polynomials in a countably infinite set of variables  $x_i$ ,  $i \geq 1$ . Let  $I$  be the ideal  $(x_1, x_2, \dots)$  of  $A$ , and set  $M := A$  and  $M' := I$ . Prove that  $M$  is a finitely generated  $A$ -module but  $M'$  is a submodule of  $M$  that is not finitely generated. Is  $M'$  a free  $A$ -module?

**A3)** Let  $A$  be a commutative ring. Suppose that  $M_1$ ,  $M_2$ , and  $N$  are submodules of an  $A$ -module  $M$  such that  $M_1 \subset M_2$ . Show that there is an exact sequence of  $A$ -modules

$$0 \rightarrow (M_2 \cap N)/(M_1 \cap N) \rightarrow M_2/M_1 \rightarrow (M_2 + N)/(M_1 + N) \rightarrow 0.$$

**A4)** (a) Prove that  $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$  and  $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$  are not isomorphic as  $\mathbb{R}$ -modules.

(b) Prove that  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$  and  $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$  are isomorphic as  $\mathbb{Q}$ -modules.

**A5)** Let  $V$  be a finite dimensional vector space over a field  $F$ . Prove that  $v \otimes w = v' \otimes w' \neq 0$  in  $V \otimes_F V$  if and only if  $v = \lambda v'$  and  $w' = \lambda w$  for some non-zero scalar  $\lambda \in F$ .

**A6)** Let  $f : A \rightarrow B$  be a ring homomorphism, let  $M$  be an  $A$ -module and  $N$  be a  $B$ -module. Let  $N_A$  be the  $A$ -module obtained from  $N$  by restriction of scalars, so that the operation of  $A$  on  $N$  is given by  $(a, n) \mapsto f(a)n$ . Show that there is a natural isomorphism

$$\mathrm{Hom}_B(B \otimes_A M, N) \cong \mathrm{Hom}_A(M, N_A).$$

**B1)** Let  $I$  be an ideal in  $\mathbb{C}[x_1, \dots, x_n]$ . Prove that  $V(I)$  is a finite set if and only if  $\mathbb{C}[x_1, \dots, x_n]/I$  is a finite dimensional complex vector space. If this occurs, show that the number of points in  $V(I)$  is at most equal to the dimension of this vector space.

**B2)** An irreducible algebraic set  $V$  in  $\mathbb{A}^n(\mathbb{C})$  is called an *affine algebraic variety*. The quotient ring  $\Gamma(V) := \mathbb{C}[x_1, \dots, x_n]/I(V)$  is the ring of polynomial functions on  $V$ , because two polynomials  $p, q \in \mathbb{C}[x_1, \dots, x_n]$  define the same function on  $V$  if and only if  $p - q \in I(V)$ . The ring  $\Gamma(V)$  is a  $\mathbb{C}$ -algebra – a commutative ring which contains  $\mathbb{C}$  as a subring.

(a) To each point  $x \in V$ , associate the ideal  $m_x$  of all  $f \in \Gamma(V)$  such that  $f(x) = 0$ . Prove that this defines a bijection between the points of  $V$  and the set of maximal ideals of  $\Gamma(V)$ .

(b) A *polynomial map*  $\phi : \mathbb{A}^n \rightarrow \mathbb{A}^m$  is a function of the form  $\phi(x) = (f_1(x), \dots, f_m(x))$  where  $f_1, \dots, f_m \in \mathbb{C}[x_1, \dots, x_n]$ . If  $V$  and  $W$  are affine algebraic varieties in  $\mathbb{A}^n$  and  $\mathbb{A}^m$ , respectively, a map  $\phi : V \rightarrow W$  is said to be *regular* if  $\phi$  is the restriction of a polynomial map  $\mathbb{A}^n \rightarrow \mathbb{A}^m$  to  $V$ . If  $f$  is a polynomial function on  $W$ , then  $f \circ \phi$  is a polynomial function on  $V$ . The induced map  $\Gamma(W) \rightarrow \Gamma(V)$  sending  $f$  to  $f \circ \phi$  is a  *$\mathbb{C}$ -algebra homomorphism* – a ring homomorphism which is also a linear map of  $\mathbb{C}$ -vector spaces. Prove that in this way, we obtain a one-to-one correspondence between the regular maps  $V \rightarrow W$  and the  $\mathbb{C}$ -algebra homomorphisms  $\Gamma(W) \rightarrow \Gamma(V)$ .

### Extra Credit Problem

**C1)** Let  $G$  be a finite group, and write  $c(G)$  for the number of distinct conjugacy classes in  $G$ . This number will increase (in general) as  $|G| \rightarrow \infty$ , so we introduce the quantity  $\gamma(G) = \frac{c(G)}{|G|}$ . The number  $\gamma(G)$  is a conjugacy class ‘density’ (it measures the average number of conjugacy classes per element of  $G$ ). Clearly,  $0 < \gamma(G) \leq 1$ , and we have  $\gamma(G) = 1$  if and only if  $G$  is abelian. From now on assume that  $G$  is *non-abelian*.

(a) Prove that  $\gamma(G) \leq \frac{5}{8}$  for every non-abelian group  $G$ .

(b) If  $p$  is the smallest prime dividing  $|G|$ , prove that  $\gamma(G) \leq \frac{1}{p} + \frac{1}{p^2} - \frac{1}{p^3}$ .

(c) Is the bound of part (a) *sharp*? That is, can you find a group  $G$  with  $\gamma(G) = \frac{5}{8}$ ? How about the bound of part (b) for groups whose orders are divisible by  $p$ ?