A1) Let $V$ be the algebraic set in $\mathbb{A}^3(\mathbb{C})$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that $V$ is a union of three irreducible components. Describe them and find their prime ideals.

A2) Let $I = (x^2 - y^3, y^2 - z^3) \subset \mathbb{C}[x, y, z]$. Define a map $\alpha : \mathbb{C}[x, y, z] \to \mathbb{C}[t]$ by $\alpha(x) = t^9$, $\alpha(y) = t^6$, $\alpha(z) = t^4$.

(a) Show that every element of $\mathbb{C}[x, y, z]/I$ is the residue of an element $a + xb + yc + yzd$, for some $a, b, c, d \in \mathbb{C}[z]$.

(b) If $f := a + xb + yc + yzd$, $a, b, c, d \in \mathbb{C}[z]$, and $\alpha(f) = 0$, compare like powers of $t$ to conclude that $f = 0$.

(c) Show that $\ker(\alpha) = I$. Deduce that $I$ is prime, $V(I)$ is irreducible, and $I(V(I)) = I$.

A3) (a) Let $F$ be an infinite field and $f \in F[x_1, \ldots, x_n]$. Suppose that $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in F$. Prove that $f = 0$.

(b) Let $f$ be a non-constant polynomial in $\mathbb{C}[x_1, \ldots, x_n]$. Show that $\mathbb{A}^n \setminus V(f)$ is infinite if $n \geq 1$, and $V(f)$ is infinite if $n \geq 2$. Conclude that the complement of any algebraic set other than $\mathbb{A}^n$ is infinite.

A4) Suppose we choose the field $F = \mathbb{R}$ and work in the real affine plane $\mathbb{A}^2(\mathbb{R})$.

(a) Show that every algebraic subset of $\mathbb{A}^2(\mathbb{R})$ is equal to $V(f)$ for some $f \in \mathbb{R}[x, y]$.

(b) Show that both the weak and strong form of Hilbert’s Nullstellensatz are false if we work with $\mathbb{R}$ instead of $\mathbb{C}$. Find a prime ideal $I$ in $\mathbb{R}[x, y]$ so that $V(I)$ is reducible.

These results indicate why we usually work with an algebraically closed field like $\mathbb{C}$ (a field $F$ is algebraically closed if any non-constant polynomial $f \in F[x]$ has a root in $F$).

A5) Let $I$ be an ideal in $\mathbb{C}[x_1, \ldots, x_n]$. Prove that $V(I)$ is a finite set if and only if $\mathbb{C}[x_1, \ldots, x_n]/I$ is a finite dimensional complex vector space. If this occurs, show that the number of points in $V(I)$ is at most the dimension of this vector space.

B1) (a) Given a finite set of maximal ideals $m_1, \ldots, m_d$ of $\mathbb{C}[x, y]$, is there a non-zero prime ideal contained in each of them? Justify your answer.

(b) Answer the same question for the ring $\mathbb{C}[x_1, \ldots, x_n]$ where $n \geq 3$.
B2) Let \( R \) be a commutative ring.

(a) Prove that an element \( x \in R \) belongs to every prime ideal of \( R \) if and only if \( x^m = 0 \) for some \( m \geq 1 \).

(b) If \( I \) is an ideal of \( R \), show that the intersection of all prime ideals \( P \) which contain \( I \) is equal to the radical of \( I \).

Note that if \( R = \mathbb{C}[x_1, \ldots, x_n] \) one can replace the word ‘prime’ above by ‘maximal’ (compare with problem B3 of Problem Set 11).

B3) A ring \( R \) is Artinian if there is no infinite decreasing chain of ideals in \( R \), i.e., whenever \( I_1 \supset I_2 \supset I_3 \supset \cdots \) is a decreasing chain of ideals in \( R \), then there is a positive integer \( m \) such that \( I_k = I_m \) for all \( k \geq m \). Suppose that \( R \) is a commutative Artinian ring. Prove as many as you can among the following:

(a) There are only finitely many maximal ideals in \( R \).
(b) The quotient \( R/J(R) \) is a direct product of a finite number of fields.
(c) There is an integer \( m \geq 1 \) such that \( (J(R))^m = (0) \).
(d) Every prime ideal of \( R \) is maximal.
(e) \( R \) is isomorphic to the direct product of a finite number of Artinian local rings.
(f) \( R \) is Noetherian.

C problem

C1) If \( A \) is a commutative Noetherian ring, prove that any surjective homomorphism \( \phi : A \to A \) is also injective. [Hint: Consider the chain of ideals \( \ker(\phi) \subseteq \ker(\phi^2) \subseteq \cdots \)]