Math 600 – Fall 2025 – Harry Tamvakis

PROBLEM SET 12 – Due December 11, 2025

A1) Let A be a commutative ring and $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. Prove that if M' and M'' are finitely generated, then M is finitely generated.

A2) Let F be a field and $A := F[x_1, x_2, \ldots]$ be the ring of polynomials in a countably infinite set of variables x_i, $i \geq 1$. Let I be the ideal $(x_1, x_2, \ldots)$ of A, and set $M := A$ and $M' := I$. Prove that M is a finitely generated A-module but M' is a submodule of M that is not finitely generated. Is M' a free A-module?

A3) Let A be a commutative ring. Suppose that M_1, M_2, and N are submodules of an A-module M such that $M_1 \subset M_2$. Show that there is an exact sequence of A-modules

$$0 \to (M_2 \cap N)/(M_1 \cap N) \to M_2/M_1 \to (M_2 + N)/(M_1 + N) \to 0.$$

A4) (a) Prove that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are not isomorphic as $\mathbb{R}$-modules.

(b) Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ are isomorphic as $\mathbb{Q}$-modules.

A5) Let V be a finite dimensional vector space over a field F. Prove that $v \otimes w = v' \otimes w' \neq 0$ in $V \otimes_F V$ if and only if $v = \lambda v'$ and $w' = \lambda w$ for some non-zero scalar $\lambda \in F$.

A6) Let $f : A \to B$ be a ring homomorphism, let M be an A-module and N be a B-module. Let N_A be the A-module obtained from N by restriction of scalars, so that the operation of A on N is given by $(a, n) \mapsto f(a)n$. Show that there is a natural isomorphism

$$\mathrm{Hom}_B(B \otimes_A M, N) \cong \mathrm{Hom}_A(M, N_A).$$

B1) Let I be an ideal in $\mathbb{C}[x_1, \ldots, x_n]$. Prove that $V(I)$ is a finite set if and only if $\mathbb{C}[x_1, \ldots, x_n]/I$ is a finite dimensional complex vector space. If this occurs, show that the number of points in $V(I)$ is at most equal to the dimension of this vector space.

B2) An irreducible algebraic set V in $\mathbb{A}^n(\mathbb{C})$ is called an *affine algebraic variety*. The quotient ring $\Gamma(V) := \mathbb{C}[x_1, \ldots, x_n]/I(V)$ is the ring of polynomial functions on V, because two polynomials $p, q \in \mathbb{C}[x_1, \ldots, x_n]$ define the same function on V if and only if $p - q \in I(V)$. The ring $\Gamma(V)$ is a $\mathbb{C}$-*algebra* – a commutative ring which contains $\mathbb{C}$ as a subring.

(a) To each point $x \in V$, associate the ideal m_x of all $f \in \Gamma(V)$ such that $f(x) = 0$. Prove that this defines a bijection between the points of V and the set of maximal ideals of $\Gamma(V)$.

(b) A *polynomial map* $\phi : \mathbb{A}^n \to \mathbb{A}^m$ is a function of the form $\phi(x) = (f_1(x), \ldots, f_m(x))$ where $f_1, \ldots, f_m \in \mathbb{C}[x_1, \ldots, x_n]$. If V and W are affine algebraic varieties in $\mathbb{A}^n$ and $\mathbb{A}^m$, respectively, a map $\phi : V \to W$ is said to be *regular* if ϕ is the restriction of a polynomial map $\mathbb{A}^n \to \mathbb{A}^m$ to V. If f is a polynomial function on W, then $f \circ \phi$ is a polynomial function on V. The induced map $\Gamma(W) \to \Gamma(V)$ sending f to $f \circ \phi$ is a $\mathbb{C}$-*algebra homomorphism* – a ring homomorphism which is also a linear map of $\mathbb{C}$-vector spaces. Prove that in this way, we obtain a one-to-one correspondence between the regular maps $V \to W$ and the $\mathbb{C}$-algebra homomorphisms $\Gamma(W) \to \Gamma(V)$.

Extra Credit Problem

C1) Let G be a finite group, and write $c(G)$ for the number of distinct conjugacy classes in G. This number will increase (in general) as $|G| \to \infty$, so we introduce the quantity $\gamma(G) = \dfrac{c(G)}{|G|}$. The number $\gamma(G)$ is a conjugacy class 'density' (it measures the average number of conjugacy classes per element of G). Clearly, $0 < \gamma(G) \leq 1$, and we have $\gamma(G) = 1$ if and only if G is abelian. From now on assume that G is *non-abelian*.

(a) Prove that $\gamma(G) \leq \dfrac{5}{8}$ for every non-abelian group G.

(b) If p is the smallest prime dividing $|G|$, prove that $\gamma(G) \leq \dfrac{1}{p} + \dfrac{1}{p^2} - \dfrac{1}{p^3}$.

(c) Is the bound of part (a) *sharp*? That is, can you find a group G with $\gamma(G) = \dfrac{5}{8}$? How about the bound of part (b) for groups whose orders are divisible by p?