Math 660 – Spring 2024 – Harry Tamvakis
PROBLEM SET 12 – Due May 9, 2024

1) The *degree* of a non-constant holomorphic map between two compact Riemann surfaces is defined to be the number of sheets of the associated branched covering map. If $f : X \to Y$ and $g : Y \to Z$ are non-constant holomorphic maps between compact Riemann surfaces, prove that the degree of $g \circ f$ is equal to $\deg(f) \deg(g)$.

2) Let X be a compact Riemann surface and let $\sigma : X \to X$ be a biholomorphic map of X onto itself, different from the identity. Let $a \in X$ be a point with $\sigma(a) \neq a$, and suppose that there is a non-constant meromorphic function f on X, holomorphic on $X \smallsetminus \{a\}$, with a pole of order k at a. Prove that σ can have at most $2k$ fixed points on X.

3) If f and g are two elliptic functions with respect to the same lattice $\Omega \subset \mathbb{C}$, prove that there exists an irreducible polynomial $P(x, y) \in \mathbb{C}[x, y]$ such that $P(f, g) = 0$.

4) For any complex number a not in the lattice Ω, prove that

$$\wp(z) - \wp(a) = -\frac{\sigma(z-a)\sigma(z+a)}{\sigma^2(z)\sigma^2(a)}.$$

5) Prove that

$$\wp'(z) = \frac{2\,\sigma(z - \frac{\omega_1}{2})\,\sigma(z - \frac{\omega_2}{2})\,\sigma(z + \frac{\omega_3}{2})}{\sigma(\frac{\omega_1}{2})\,\sigma(\frac{\omega_2}{2})\,\sigma(-\frac{\omega_3}{2})\,\sigma(z)^3}\,.$$

Extra Credit Problems.

Let $\Omega \subset \mathbb{C}$ be a lattice and $\wp(z)$ the associated Weierstrass $\wp$-function. We have seen that $\wp(z)$ satisfies the differential equation $(\wp'(z))^2 = p(\wp(z))$, where $p(x) = 4x^3 - g_2 x - g_3$. The following three problems examine the conditions under which the coefficients g_2 and g_3 of $p(x)$ are real numbers.

EC1) Prove that the following conditions are equivalent: (i) $g_2, g_3 \in \mathbb{R}$; (ii) $G_k \in \mathbb{R}$ for all $k \geq 3$; (iii) $\wp(\overline{z}) = \overline{\wp(z)}$ for all $z \in \mathbb{C}$; (iv) $\overline{\Omega} = \Omega$ (the last condition says that Ω is a *real lattice*).

EC2) We say that Ω is *real rectangular* if $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ where $\omega_1 \in \mathbb{R}$ and $\omega_2 \in i\mathbb{R}$, and that Ω is *real rhombic* if $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ where $\omega_2 = \overline{\omega}_1$. Prove that a lattice Ω is real if and only if it is real rectangular or real rhombic.

EC3) Let Ω be a real lattice. Define the real elliptic curve $E_{\mathbb{R}}$ to be the set $\{(x, y) \in \mathbb{R}^2 \mid y^2 = p(x)\}$. Prove that $E_{\mathbb{R}}$ has one or two connected components as Ω is real rhombic or real rectangular, respectively.