Math 600 – Fall 2025 – Harry Tamvakis
PROBLEM SET 2 – Due September 18, 2025

A1) (a) Let A and B be $m \times n$ and $k \times n$ matrices, respectively, with entries in a field F. We say that A is *right divisible* by B if there exists an $m \times k$ matrix C such that $A = CB$. If A is right divisible by B, then clearly $Bx = 0$ implies $Ax = 0$ for any column vector $x \in F^n$. Is the converse true?

(b) What is the analogous necessary condition to the one in (a) for 'left divisibility'? Is this condition sufficient?

A2) Let $T : V \to V$ be a linear operator on a real vector space V such that $T^2 = I$. Define subspaces W^+ and W^- of V as follows:

$$W^+ = \{v \in V \mid T(v) = v\} \qquad W^- = \{v \in V \mid T(v) = -v\}.$$

Prove that V is isomorphic to the direct sum $W^+ \oplus W^-$.

A3) Let U and V be subspaces of a vector space W and suppose that $U \subset V$. The quotient space V/U is a subspace of W/U in a natural way. Prove that

$$(W/U)/(V/U) \cong W/V.$$

A4) Let A be an invertible matrix. Prove that if the rank of the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is equal to the rank of A, then $D = CA^{-1}B$.

B1) In this problem, V is a finite dimensional vector space over a field F, and $T : V \to V$ is any linear transformation. Write K_n for $\mathrm{Ker}(T^n)$ and J_n for $\mathrm{Im}(T^n)$. Both K_n and J_n are subspaces of V for each n.

(a) Show that $K_n \subset K_{n+1}$ and $J_{n+1} \subset J_n$ for each n. Prove that there exists an N such that for all $n \geq N$, $K_n = K_{n+1}$ and $J_n = J_{n+1}$.

(b) Write K for the common subspace K_n when $n \geq N$ and similarly write J for the common subspace J_n for $n \geq N$. Prove that $V = J \oplus K$.

(c) Show that $T(J) \subset J$ and $T(K) \subset K$.

(d) Show that there exists an r such that $T^r(x) = 0$ for all $x \in K$ and further show that T maps J *isomorphically onto itself*.

Thus: Given a finite dimensional vector space V and a linear transformation $T : V \to V$, the vector space V splits into a direct sum of two pieces; one where T is an *isomorphism* and the other where T is *nilpotent*.

(e) Suppose you know that V admits a basis $v_1, \ldots, v_n$ such that $T(v_i) = \lambda_i v_i$, for some $\lambda_i \in F$, $1 \leq i \leq n$. In terms of the vectors v_i, and corresponding to the values λ_i, describe J and K for this T.

B2) (a) Let V be any vector space over a field F. Suppose we are given a rule which associates to each *subset*, S, of V a *subspace*, $[S]$ of V, and we are told that this rule obeys the laws:

(1) For each S, S is contained in $[S]$;

(2) If $S \subset T$, then $[S] \subset [T]$;

(3) For each S, we have $[S] = [[S]]$;

(4) If W is any subspace of V and $W \neq V$, then $[W] \neq V$.

Prove that for each subset S, $[S] = \mathrm{Span}(S)$. (Note: It is *not* assumed that V is finite dimensional. If you must assume this – partial credit.)

(b) Provide counterexamples to show that the result of part (a) is false if we remove either of the conditions (1) or (4). What about (2) or (3)?

Extra Credit Problem

C1) An infinite set A is *countable* if there is a bijection from A to the set $\mathbb{N}$ of natural numbers. Let F be any field and V be the F-vector space of all infinite sequences $\{a_n\}_{n \geq 1}$ of elements in F, with the usual pointwise operations. V is isomorphic to the direct product $\prod_{n=1}^{\infty} F$. Prove that V does not have a countable basis.