Math 601 – Spring 2026 – Harry Tamvakis

PROBLEM SET 2 – Due February 19, 2026

A1) Suppose $H \subset \mathbb{Z}^n$ is a subgroup generated by n linearly independent vectors $v_1, \ldots, v_n$ in $\mathbb{Z}^n$, and let $|\mathbb{Z}^n : H|$ be the index of H in $\mathbb{Z}^n$. Prove that

$$|\mathbb{Z}^n : H| = |\det(v_1, \ldots, v_n)|.$$

A2) If A and B are finitely generated abelian groups, show that $A \otimes_{\mathbb{Z}} B$ is a finitely generated abelian group. Determine its rank and elementary divisors in terms of those of A and B.

A3) Suppose $T : V \to V$ is a linear operator on a finite dimensional vector space V over a field F.

(a) Show that T is similar to an upper triangular matrix if and only if the minimal polynomial of T is a product of linear factors over F.

(b) How can one determine, from the minimal polynomial of T, whether or not T is diagonalizable? Justify your answer.

(c) If $F = \mathbb{C}$, express the minimal polynomial and characteristic polynomial of T in terms of a Jordan canonical matrix for T.

A4) Consider the three matrices in $M_4(\mathbb{C})$:

$$A = \begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Find the Jordan canonical forms of A, B, C and determine which ones are similar.

B1) (a) Suppose R is a P.I.D. and $\phi : R^n \to R^m$ a non-zero R-linear map. Show that there is a basis $x_1, \ldots, x_n$ of R^n and a basis $y_1, \ldots, y_m$ of R^m and an integer $r \geq 0$ such that $\phi(x_i) = d_i y_i$, $1 \leq i \leq r$, $d_r \neq 0$ and $\phi(x_j) = 0$ for $r + 1 \leq j \leq n$. Furthermore, we have $d_1 | d_2 | \cdots | d_r$.

(b) Define $GL_k(R)$ to be the units of the ring $M_k(R)$ of $k \times k$ matrices with entries in R. Two $m \times n$ matrices A and B with entries in R are called *equivalent* if there is a $P \in GL_m(R)$ and a $Q \in GL_n(R)$ such that $B = PAQ$. Show that when R is a P.I.D., every matrix is equivalent to one of the form

$$\begin{pmatrix} d_1 & & 0 & \dots & 0 \\ & d_2 & & & \\ 0 & & \ddots & & \vdots \\ \vdots & & & d_r & \\ 0 & & \dots & & 0 \end{pmatrix}$$

for some non-zero elements $d_1, \dots, d_r$ in R such that $d_1|d_2|\cdots|d_r$, with $r \le \min(m, n)$. Show that r is unique, and $d_1, \dots, d_r$ are unique up to multiplication by units. [Don't redo work done in class.]

(c) If F is a field, prove that the matrices $A, B \in M_n(F)$ are similar if and only if the matrices $xI - A$ and $xI - B$ are equivalent in $M_n(F[x])$.

B2) Find and prove a canonical form for real square matrices, where the blocks have the form

$$\begin{pmatrix} A & I & & & \\ & A & I & & \\ & & A & & \\ & & & \ddots & I \\ & & & & A \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a & 1 & & & \\ & a & 1 & & \\ & & a & & \\ & & & \ddots & 1 \\ & & & & a \end{pmatrix}$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, A is a 2×2 matrix of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, and a, b are real numbers.

B3) Suppose that we are given two fields K and L with $K \subset L$. Write $M_n(K)$ (respectively $M_n(L)$) for the $n \times n$ matrices with entries in K (respectively L), and choose $A, B \in M_n(K)$.

(a) If there exists a matrix $Q \in GL_n(L)$ such that $B = QAQ^{-1}$, show that there is a $P \in GL_n(K)$ such that $B = PAP^{-1}$.

(b) If $B = A^t$, show that such a matrix P always exists.

Extra Credit Problem

C1) Suppose V is a finite dimensional complex vector space and T a linear operator on V. Prove that T can be written in a *unique* way as a sum $T = S + N$, where S is diagonalizable, N is nilpotent, and $SN = NS$.