Math 606 – Fall 2026 – Harry Tamvakis
PROBLEM SET 2 – Due September 17, 2026

1) Suppose that $p_1, \dots, p_n$ are points on the compact Riemann surface X and $X' = X \smallsetminus \{p_1, \dots, p_n\}$. Consider a non-constant holomorphic function $f : X' \to \mathbb{C}$. Show that the image of f comes arbitrarily close to every complex number c.

2) (a) Prove that $\tan : \mathbb{C} \to \mathbb{P}^1$ is a local homeomorphism.

(b) Show that $\tan(\mathbb{C}) = \mathbb{P}^1 \smallsetminus \{-i, i\}$ and

$$\tan : \mathbb{C} \to \mathbb{P}^1 \smallsetminus \{-i, i\}$$

is a covering map.

3) Suppose that X is a compact Riemann surface such that there exists a meromorphic function on X having exactly one pole of order 1. Prove that X is biholomorphic to the Riemann sphere.

4) Determine the branch points of the map $f : \mathbb{C} \to \mathbb{P}^1$ with

$$f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

5) The *degree* of a non-constant holomorphic map between two compact Riemann surfaces is defined to be the number of sheets of the associated branched covering map. If $f : X \to Y$ and $g : Y \to Z$ are non-constant holomorphic maps between compact Riemann surfaces, prove that the degree of $g \circ f$ is equal to $\deg(f)\deg(g)$.

6) Let X be a compact Riemann surface and let $\sigma : X \to X$ be a biholomorphic map of X onto itself, different from the identity. Let $a \in X$ be a point with $\sigma(a) \neq a$, and suppose that there is a non-constant meromorphic function f on X, holomorphic on $X \smallsetminus \{a\}$, with a pole of order k at a. Prove that σ can have at most $2k$ fixed points on X.