Math 660 – Spring 2024 – Harry Tamvakis
PROBLEM SET 2 – Due February 15, 2024

1) (a) Let $f : \mathbb{C} \to \mathbb{C}$ and $g : \mathbb{C} \to \mathbb{C}$ be holomorphic functions and suppose that $|f(z)| \leq |g(z)|$, for all $z \in \mathbb{C}$. What conclusion can you draw? Justify your answer.

(b) Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that there exist positive real numbers A, B and $n \in \mathbb{N}$ with $|f(z)| \leq A|z|^n + B$ for every $z \in \mathbb{C}$. Prove that f is a polynomial.

2) Let c be a positive real number.

(a) Suppose that $f \in \mathcal{O}(\mathbb{C})$ is non-constant. Show that the closure of the set $\{z : |f(z)| < c\}$ is the set $\{z : |f(z)| \leq c\}$.

(b) Let p be a non-constant polynomial. Show that every connected component of the set $\{z : |p(z)| < c\}$ contains a zero of p.

3) Let $D = \{z \in \mathbb{C} \ : \ |z| < 1\}$, let $f \in \mathcal{O}(D)$ and suppose that $|f(z)| \to 1$ as $|z| \to 1$. Prove that there exist finitely many points $a_1, \ldots, a_n \in D$ and integers $m_1, \ldots, m_n$ such that

$$f(z) = e^{i\theta} \prod_{j=1}^{n} \left(\frac{z - a_j}{1 - \overline{a}_j z} \right)^{m_j}$$

where θ is a real constant. [Hint: Show that if, in addition, f has no zeros, then f is constant.]

4) Let $\Omega \subset \mathbb{C}$ be open and $f \in \mathcal{O}(\Omega)$. Prove that $f^{-1}(\mathbb{R})$ cannot be a non-empty *compact* subset of Ω.

5) Suppose that Ω is a bounded open set in $\mathbb{C}$, $\{f_n\}$ is a sequence of continuous functions on $\overline{\Omega}$ which are holomorphic on Ω, and $\{f_n\}$ converges uniformly on the boundary of Ω. Prove that $\{f_n\}$ converges uniformly on $\overline{\Omega}$.