Math 406 – Fall 2025 – Harry Tamvakis

PROBLEM SET 3 – Due September 25, 2025

Reading for this week: Section 2.

Problems

Section 2, Problems #2, 3, 10, 12. In addition, do the following problems:

A1) (a) Prove that $(a, b) = (a, c) = 1$ implies $(a, bc) = 1$.

(b) Prove that $(a, b) = 1$ and $c|a$ imply $(c, b) = 1$.

A2) Bob says 'I've been looking for half an hour for a number n such that n and $n + 20$ have greatest common divisor of 7 and I haven't found one. I think I'll program it on the computer.' His sister Helen says, 'The computer won't find one, either.' How did Helen know that?

A3) (a) Prove that if the positive integer n is a square, then each exponent in the prime-power decomposition of n is even.

(b) Conversely, prove that if each exponent in the prime-power decomposition of n is even, then n is a square.

A4) (a) In 1511, Carolus Bouvellus claimed that for each $n \geq 1$, one or both of $6n - 1$ and $6n + 1$ were prime. Show that this conjecture is false.

(b) Bouvellus must have realized something was amiss because he soon revised his claim to read that every prime, except 2 and 3, can be expressed in the form $6n \pm 1$, for some natural number n. Show that this conjecture is true.

(c) Prove that $\{3, 5, 7\}$ is the only set of three consecutive odd numbers that are all prime.

A5) Recall that $n!$ (pronounced *n factorial*) denotes the product $1 \cdot 2 \cdots n$ of all natural numbers from 1 to n. Prove that if $n > 4$ is composite, then n

divides $(n-1)!$. Conversely, show that if n is prime, then n does *not* divide $(n-1)!$.

A6) Find all prime numbers p such that $17p+1$ is a perfect square.

Extra Credit Problems.

EC1) Let $n \geq 1$ be an integer. Prove that if you choose $n+1$ positive integers less than or equal to $2n$, then there must be two of them which are relatively prime.

EC2) Let m and n be positive integers and suppose that a is an integer greater than 1. Use the Euclidean algorithm to prove that

$$\gcd(a^m - 1, a^n - 1) = a^{\gcd(m,n)} - 1.$$