Math 600 – Fall 2025 – Harry Tamvakis

PROBLEM SET 3 – Due September 25, 2025

A1) Although there is no canonical way to identify a vector space V with its dual V^*, there is a canonical way to associate, to any linear map $T : V \to W$, a linear map $T^* : W^* \to V^*$ called the *adjoint* of T. The adjoint function T^* is defined by $T^*(g)(v) = g(T(v))$, for any $g \in W^*$ and $v \in V$. Suppose that $\mathcal{B} = (e_1, \ldots, e_n)$ and $\mathcal{C} = (f_1, \ldots, f_m)$ are ordered bases of V and W, respectively, and let $A = (T)_{\mathcal{B},\mathcal{C}}$ be the matrix of $T : V \to W$ with respect to these bases. Show that matrix of the adjoint map T^* with respect to the dual bases $\mathcal{C}^*$ and $\mathcal{B}^*$ is the transpose matrix A^t.

A2) Let V be a finite dimensional vector space and let $\mathcal{B} = (e_1, \ldots, e_n)$ be an ordered basis for V. The basis $\mathcal{B}$ induces a linear isomorphism $V \to V^*$ by sending e_i to e_i^* for each i. If $\mathcal{C}$ is another ordered basis of V, prove that $\mathcal{B}$ and $\mathcal{C}$ induce the *same* isomorphism $V \to V^*$ if and only if the change of basis matrix A from $\mathcal{B}$ to $\mathcal{C}$ satisfies $AA^t = I$ (if V is a real vector space, then this last condition means that A is an *orthogonal* matrix).

A3) Let $A \in M_n(\mathbb{C})$ have n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Assume that $|\lambda_1| > |\lambda_i|$ for $2 \le i \le n$. Prove that for most vectors $x \in \mathbb{C}^n$, the sequence $x_k = \lambda_1^{-k} A^k x$ converges in $\mathbb{C}^n$ to an eigenvector y with eigenvalue λ_1. Describe precisely the conditions x must satisfy for this to be the case.

B1) We say that a sequence of vector spaces and linear maps

$$U \xrightarrow{\alpha} V \xrightarrow{\beta} W$$

is *exact at* V if $\mathrm{Im}(\alpha) = \mathrm{Ker}(\beta)$. A sequence $\cdots \to V_{n-1} \to V_n \to V_{n+1} \to \cdots$ of linear maps between vector spaces V_i is called an *exact sequence* if it is exact at every V_i which lies between a pair of arrows.

(a) Prove that if

$$0 \longrightarrow U \xrightarrow{\alpha} V \xrightarrow{\beta} W \longrightarrow 0$$

is an exact sequence of vector spaces, then $V \cong U \oplus W$. Furthermore, prove that the induced sequence

$$0 \longrightarrow W^* \xrightarrow{\beta^*} V^* \xrightarrow{\alpha^*} U^* \longrightarrow 0$$

is exact (see problem A1 for the definition of α^* and β^*).

(b) Prove that if $0 \to V_0 \to V_1 \to \cdots \to V_n \to 0$ is an exact sequence of finite dimensional vector spaces, then $\displaystyle\sum_{i=0}^{n} (-1)^i \dim(V_i) = 0$.

B2) Let V be an n-dimensional real vector space. Assume that there is a linear map $J : V \to V$ such that $J^2 = -I$, where I denotes the identity map on V. The pair (V, J) is called an *almost complex vector space.*

(a) Prove that n must be even.

(b) Suppose that $n = 2k$. Show that V has an ordered basis

$$\mathcal{B} = (u_1, v_1, u_2, v_2, \ldots, u_k, v_k)$$

such that the matrix of J in this basis is

$$(J)_{\mathcal{B}} = \begin{pmatrix} 0 & -1 & & & & & & \\ 1 & 0 & & & & & & \\ & & 0 & -1 & & & & \\ & & 1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & -1 \\ & & & & & & 1 & 0 \end{pmatrix}.$$

(c) If A is any $n \times n$ real or complex matrix, the *matrix exponential*

$$e^A := I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots \tag{1}$$

is the square matrix obtained by substituting A into the Taylor series for e^x. One can show that the series (1) converges (absolutely) in $M_n(\mathbb{C})$ (you may assume this). Check that if P is any invertible $n \times n$ matrix, then

$$Pe^A P^{-1} = e^{PAP^{-1}}. \tag{2}$$

Show how (2) allows us to define the exponential e^T of any linear map $T : V \to V$. If θ is a real number, compute $e^{\theta J}$ explicitly in terms of J and real valued functions of θ. Finally, evaluate $\det(e^{\theta J})$.

B3) In this problem, I expect a solution that uses only the material that we have covered so far in class. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map.

(a) Suppose T has at least one real eigenvalue λ. Show that the matrix for T in a suitable basis is one of

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad \text{where } \lambda, \mu \in \mathbb{R}.$$

(b) Now suppose T has no real eigenvalues. Show that T may be put into the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \text{where } \lambda, \mu > 0.$$

Extra Credit Problem

C1) Let $A = \{a_{ij}\}_{1 \le i,j \le n}$ and $B = \{b_{ij}\}_{1 \le i,j \le n}$ be two matrices in $M_n(\mathbb{C})$ and $\delta > 0$. We say that A and B are *δ-close* if $|a_{ij} - b_{ij}| < \delta$ for each $i, j \in [1, n]$. This defines a topology on $M_n(\mathbb{C})$.

(a) Prove that the eigenvalues of a matrix in $M_n(\mathbb{C})$ depend continuously on its entries. More precisely, let $A \in M_n(\mathbb{C})$ be a given matrix. Show that for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $B \in M_n(\mathbb{C})$ is δ-close to A and λ is an eigenvalue of A, then there exists an eigenvalue μ of B such that $|\lambda - \mu| < \epsilon$.

(b) Prove that the diagonalizable matrices form a dense subset of $M_n(\mathbb{C})$. More precisely, show that for any matrix A in $M_n(\mathbb{C})$ and $\delta > 0$, there exists a diagonalizable matrix B that is δ-close to A.