**Math 601 – Fall 2026 – Harry Tamvakis**

**PROBLEM SET 3 – Due February 26, 2026**

**A1)** Prove that every real symmetric matrix has a real symmetric cube root; i.e., if $A$ is real symmetric, there is a real symmetric $B$ such that $B^3 = A$. Characterize those real symmetric matrices that have a real symmetric square root.

**A2)** Find all values of the real number $a$ for which the quadratic form

$$Q(x, y, z) = x^2 + y^2 + z^2 + 2axy + 2ayz$$

is positive definite.

**A3)** Let $V$ be a finite dimensional vector space and $B : V \times V \to F$ be a non-degenerate alternating bilinear form on $V$. In class we proved that $\dim(V) = 2n$ is even. A subspace $W$ of $V$ is called *isotropic* if $B(x, y) = 0$ for all $x, y \in W$. Prove that if $W$ is isotropic then $0 \leq \dim W \leq n$. Give an example of an isotropic subspace $W$ with $\dim(W) = n$.

**B1)** Let $T$ be a normal operator on a finite dimensional complex inner product space $V$. Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of $T$. Let $W_i$ be the eigenspace of $\lambda_i$ and $E_i : V \to V$ the orthogonal projection of $V$ onto $W_i$. You assume the spectral theorem for $T$, which was proved in class.

(a) Prove that we have

$$T = \lambda_1 E_1 + \cdots + \lambda_k E_k = \sum_{i=1}^{k} \lambda_i E_i. \tag{1}$$

The decomposition (1) is called the *spectral resolution* of $T$.

(b) The orthogonal projections $E_i$ in (1) are canonically associated with $T$; in fact, they are polynomials in $T$. Show that if we define polynomials $e_i$ by

$$e_i(x) = \prod_{j \neq i} \left( \frac{x - \lambda_j}{\lambda_i - \lambda_j} \right),$$

then $E_i = e_i(T)$ for $1 \leq i \leq k$.

(c) Let $S = \{\lambda_1, \ldots, \lambda_k\}$ be the set of eigenvalues of $T$; $S$ is called the *spectrum* of $T$. If $f : S \to \mathbb{C}$ is any function, the linear operator $f(T)$ is defined by the equation

$$f(T) := \sum_{i=1}^{k} f(\lambda_i) E_i.$$

Prove that $f(T)$ is a normal operator whose spectrum is $f(S)$.

(d) Suppose that $S : V \to V$ is a linear operator such that $ST = TS$. Prove that $ST^* = T^*S$.

**B2)** Let $V$ be a finite dimensional vector space over a field $F$ and suppose that $B : V \times V \to F$ is a bilinear form on $V$. We say that $B$ is *symmetric* if $B(x, y) = B(y, x)$ for any $x, y \in V$ and $B$ is *alternating* if $B(x, x) = 0$ for any $x \in V$. Assume that whenever $x, y \in V$ are such that $B(x, y) = 0$, then $B(y, x) = 0$ (that is, the orthogonality relation on $V$ induced by $B$ is symmetric). Prove that either $B$ is symmetric or $B$ is alternating.

**B3)** Let $V$ be a vector space of dimension $2n$ over a field $F$ and choose a basis $e_1, \dots, e_{2n}$ of $V$. To each skew-symmetric matrix $A = \{a_{ij}\} \in M_{2n}(F)$ assign the element $\omega := \sum_{i<j} a_{ij}\, e_i \wedge e_j$ in $\Lambda^2 V$, and then to $\omega$ assign $\wedge^n \omega = f(A)\, e_1 \wedge \cdots \wedge e_{2n}$ in $\Lambda^{2n} V$. The function $A \mapsto f(A)$ does not depend on the choice of basis of $V$. Suppose that $e'_1, \dots, e'_{2n}$ is a different basis with $e'_j = \sum_i p_{ij} e_i$, and let $P$ be the invertible matrix $\{p_{ij}\}$.

(a) Show that $\sum_{i<j} a_{ij}\, e_i \wedge e_j = \sum_{i<j} b_{ij}\, e'_i \wedge e'_j$ where the matrix $B := \{b_{ij}\}$ satisfies $A = PBP^t$. Deduce that $f(PBP^t) = (\det P) f(B)$.

(b) If $A$ is invertible, show that $A = PJP^t$ for some matrix $P$, where $J := \operatorname{diag}(S, \dots, S)$ with $S := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Deduce that $f(A) = (\det P) f(J)$ and $\det A = (\det P)^2 = (f(A)/f(J))^2$.

(c) Prove that $f(A) = n! \sum_I \operatorname{sgn}(I)\, a_{i_1 i_2} a_{i_3 i_4} \cdots a_{i_{2n-1} i_{2n}}$ summed over all partitions $I$ of the set $\{1, \dots, 2n\}$ into $n$ pairs $\{i_k, i_{k+1}\}$ where $i_k < i_{k+1}$ and $\operatorname{sgn}(I)$ is the sign of the permutation $\begin{pmatrix} 1 & 2 & \cdots & 2n \\ i_1 & i_2 & \cdots & i_{2n} \end{pmatrix}$. In particular, $f(J) = n!$. Conclude with an explicit formula for $\operatorname{Pfaffian}(A) := f(A)/f(J)$.

## <u>Extra Credit Problems</u>

**C1)** Suppose that $A = \{a_{ij}\}_{1 \le i,j \le n}$ is a symmetric matrix in $M_n(\mathbb{R})$. Prove that $A$ is positive definite if and only if for each $m \in [1, n]$, we have

$$\det\left(\{a_{ij}\}_{1 \le i,j \le m}\right) > 0.$$

**C2)** Let $A$ be a symmetric positive definite $n \times n$ real matrix, $x := (x_1, \dots, x_n)$, and $\langle\ ,\ \rangle$ denote the usual inner product on $\mathbb{R}^n$. Evaluate the integral

$$\int_{\mathbb{R}^n} e^{-\langle Ax, x\rangle}\, dx_1 \cdots dx_n.$$