Math 660 – Spring 2024 – Harry Tamvakis

PROBLEM SET 3 – Due February 22, 2024

1) Exercise 96 from the textbook.

2) Exercise 113 from the textbook.

3) (a) Let $D = D(a, R) := \{z \in \mathbb{C} : |z - a| < R\}$. Let $f \in \mathcal{O}(D)$, and suppose that $|f(z)| \le M$, for all $z \in D$. Prove that if $f^{(n)}(a) = 0$ for $0 \le n < k$, then

$$|f(z)| \le M \left(\frac{r}{R}\right)^k$$

for $|z - a| \le r < R$.

(b) Let Ω be a connected open set in $\mathbb{C}$ and let $H^\infty(\Omega)$ denote the $\mathbb{C}$-vector space of bounded holomorphic functions on Ω. Set $\|f\|_\infty := \sup_{z \in \Omega} |f(z)|$ for $f \in H^\infty(\Omega)$. Prove that if $K \subset \Omega$ is compact and $\epsilon > 0$, there exists a subspace $V \subset H^\infty(\Omega)$ of *finite codimension* such that

$$|f(z)| \le \epsilon \|f\|_\infty, \quad \forall z \in K, \quad \forall f \in V.$$

4) Suppose that X and Y are connected topological manifolds.

(a) Let $p : X \to Y$ be a covering map. Suppose that there is a $y_0 \in Y$ such that $p^{-1}(y_0)$ contains exactly n points, for some $n \in \mathbb{N}$. Prove that $p^{-1}(y)$ contains exactly n points for every $y \in Y$.

(b) If $p : X \to Y$ is a local homeomorphism such that there is an integer $n \ge 1$ with $\#p^{-1}(y) = n$ for all $y \in Y$, prove that p is a covering map.

(c) Let $p : X \to Y$ be a local homeomorphism and suppose that p is a *proper* map (i.e., that $p^{-1}(K)$ is compact for any compact set $K \subset Y$). Prove that p is a covering map.

5) (a) Prove that the germs defined by z and by $1/z$ at $z = 1$ lie in different connected components of $\mathcal{O}$.

(b) Let $f, g : \mathbb{C} \to \mathbb{C}$ be entire functions and suppose that the germs f_0 and g_0 defined by f and g respectively at zero lie in the same connected component of $\mathcal{O}$. Prove that $f = g$.

Extra Credit Problem.

EC) (a) Let $\{a_n\}_{n\geq 0}$ be a sequence of real numbers. Prove that there is a C^∞ function f defined on an open neighborhood of 0 in $\mathbb{R}$ such that $f^{(n)}(0) = a_n$ for every $n \geq 0$.

[Hint: Fix $n \geq 1$, $c \in \mathbb{R}$, and $\epsilon > 0$. Construct a C^∞ function φ such that $\varphi(x) = cx^n$ for x near 0 and such that $\sup_{x\in\mathbb{R}} |\varphi^{(k)}(x)| \leq \epsilon$ for all $k < n$. For each n, choose c and ϵ appropriately and add the resulting functions φ together.]

(b) Give an example to show that the statement in part (a) is false for holomorphic functions f defined on an open neighborhood of 0 in $\mathbb{C}$.