Math 600 – Fall 2025 – Harry Tamvakis
PROBLEM SET 4 – Due October 2, 2025

In the following problems I expect solutions that use only the material that we have covered so far in class.

A1) Let $a_1, \ldots, a_n$ be real numbers not all zero and H denote the hyperplane in $\mathbb{R}^n$ with equation $a_1x_1 + \cdots + a_nx_n = 0$. Prove that the distance d from a point $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ to the hyperplane H is given by

$$d = \frac{|a_1v_1 + a_2v_2 + \cdots + a_nv_n|}{\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}}.$$

A2) Let $\mathbb{R}[x]$ be the vector space of all polynomials with real coefficients, equipped with the inner product

$$\langle f, g\rangle := \int_{-1}^{1} f(x)g(x)\,dx.$$

(a) Prove that there is *unique* orthogonal basis $\{f_n\}_{n\geq 0}$ of $\mathbb{R}[x]$ such that

(i) f_n has degree n for each $n \geq 0$ and (ii) $f_n(1) = 1$ for each $n \geq 0$.

(b) Compute the polynomials f_0, f_1, f_2, and f_3.

A3) Let V be a complex vector space with a hermitian inner product. Find a formula which expresses $\langle u, v\rangle$ in terms of $\|u+v\|^2$, $\|u-v\|^2$, $\|u+iv\|^2$ and $\|u-iv\|^2$, for any vectors $u, v \in V$.

B1) Let V and W be finite dimensional complex vector spaces equipped with hermitian inner products.

(a) If f is any linear functional in V^*, prove that there exists a *unique* vector $w \in V$ such that $f(v) = \langle v, w\rangle$ for all $v \in V$.

(b) Deduce from (a) that there is a conjugate linear isomorphism $V \to V^*$ that does not depend on the choice of any basis of V.

(c) Let $T : V \to W$ be a linear map. Show that there is a unique linear map $T' : W \to V$ such that $\langle T(x), y\rangle = \langle x, T'(y)\rangle$ for any $x \in V$ and $y \in W$. How is the map T' related to the adjoint map $T^* : W^* \to V^*$ defined in problem A1 of last week's homework set?

B2) (a) Suppose A is an $m \times n$ matrix and $b \in \mathbb{R}^m$. In case the equation $Ax = b$ has no solution, it is often desirable to find the next best thing, an $\widehat{x}$ that makes $A\widehat{x}$ as close as possible to b. Call $\widehat{x}$ a *nearest solution* if

$$\|A\widehat{x} - b\| \leqslant \|Ax - b\|$$

for all x in $\mathbb{R}^n$ (where the norm is the usual one coming from the dot product). Prove that $\widehat{x}$ is a nearest solution of $Ax = b$ if and only if $A^tA\widehat{x} = A^tb$.

(b) Show that A^tA is invertible if and only if A has linearly independent columns. In this case there is a unique nearest solution of $Ax = b$, namely $\widehat{x} = (A^tA)^{-1}A^tb$.

(c) Suppose that we are given points $(x_1, y_1), \ldots, (x_n, y_n)$ in the Euclidean plane $\mathbb{R}^2$. A line $y = \alpha x + \beta$ passes through each point exactly when the system

$$\alpha x_i + \beta = y_i, \qquad 1 \leqslant i \leqslant n$$

has a solution in α and β. If there are more than two points then this system may not have a solution. What condition on the points guarantees that there is a unique nearest solution $(\widehat{\alpha}, \widehat{\beta})$ for the system? In the latter case derive formulas for $\widehat{\alpha}$ and $\widehat{\beta}$ in terms of the x_i's and y_i's.

B3) Let V be a finite dimensional vector space equipped with a hermitian inner product $\langle\ ,\ \rangle$. Suppose that $\{u_1, \ldots, u_m\}$ and $\{v_1, \ldots, v_m\}$ are two sets of m vectors in V. Prove that the following statements are equivalent.

(a) There exists a unitary linear map $T : V \to V$ such that $T(u_i) = v_i$ for $i = 1, \ldots, m$.

(b) The $m \times m$ matrices $\{\langle u_i, u_j\rangle\}_{1\leq i,j\leq m}$ and $\{\langle v_i, v_j\rangle\}_{1\leq i,j\leq m}$ are equal.

B4) Let x be a unit column vector in $\mathbb{R}^n$.

(a) Prove that the matrix $A := I - 2xx^t$ is orthogonal.

(b) Show that left multiplication by A is a *reflection* through the hyperplane $H := \langle x\rangle^\perp$ which is orthogonal to x. That is, prove that if we write an arbitrary vector v in the form $v = \lambda x + y$, where $\lambda \in \mathbb{R}$ and $y \in H$, then $Av = -\lambda x + y$.

(c) Suppose that $u, v \in \mathbb{R}^n$ are arbitrary vectors in $\mathbb{R}^n$ of the same length. Determine a unit vector x such that $Au = v$.

(d) Prove that every orthogonal $n \times n$ matrix is a product of at most n reflections.