**Math 601 – Spring 2026 – Harry Tamvakis**

**PROBLEM SET 4 – Due March 5, 2026**

**A1)** Verify that the characteristic polynomial of the matrix

$$A := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -2 & 0 & 1 \\ -2 & 0 & -1 & -2 \end{pmatrix}$$

is a product of linear factors in $\mathbb{Q}[x]$. Determine the rational and Jordan canonical forms for $A$ in $M_n(\mathbb{Q})$.

**A2)** If $P$ is a projective $R$-module, prove that there is a free $R$-module $F$ such that $F \cong F \oplus P$.

**A3)** Suppose that $P$ and $P'$ are projective modules over a commutative ring $R$ and that $0 \to K \to P \to M \to 0$ and $0 \to K' \to P' \to M \to 0$ are two short exact sequences of $R$-modules. Prove that $K \oplus P' \cong K' \oplus P$. [Hint: If $\phi : P \to M$ and $\phi' : P' \to M$, consider the submodule of $P \oplus P'$ consisting of those pairs $(x, y)$ such that $\phi(x) = \phi'(y)$.]

**A4)** Let $R$ be a commutative ring. In this problem, an $R$-*algebra* means a commutative ring $A$ together with a homomorphism $\phi : R \to A$. We usually omit $\phi$ from the notation and write $ra$ for $\phi(r)a$ when $r \in R$ and $a \in A$. Suppose that for each $R$-algebra $A$ we are given a map of sets $f_A : A \to A$ such that for every $R$-algebra homomorphism $g : A \to B$ we have $g \circ f_A = f_B \circ g$. Prove that there exists a polynomial $p \in R[x]$ such that for every $R$-algebra $A$ and $a \in A$, we have $f_A(a) = p(a)$. [Hint: One way to do this uses Yoneda's lemma.]

**B1)** Problem 24 in Chapter 10.3 of Dummit and Foote (page 358). This proves the rather interesting fact that the direct product of infinitely many copies of $\mathbb{Z}$ is not a free $\mathbb{Z}$-module.

**B2)** Suppose that $R$ is a principal ideal domain.

(a) Let $F$ be a free $R$-module and $E$ a submodule of $F$. Prove that $E$ is also free. [Hint: We did this in class when $F$ has a finite basis. In general, you will need to use Zorn's Lemma. One suggestion is to let $B$ be a basis of $F$ and consider the set of pairs $(A, L)$ such that $A \subset B$ and $R\langle A\rangle \cap E$ is free with basis $L$, ordered by inclusion in the natural way.]

(b) Show that any projective $R$-module is free. It follows from this and Problem B1 that the direct product of projective modules need not be projective.

**B3)** Let $R$ be a commutative ring and $P$ be a finitely generated projective $R$-module.

(a) Write $P^*$ for the dual module $\mathrm{Hom}_R(P, R)$. Prove that the natural map $P \to P^{**}$ is an isomorphism of $R$-modules.

(b) Let $\mathrm{End}(P) = \mathrm{Hom}_R(P, P)$. Show that there is a natural isomorphism $P^* \otimes P \cong \mathrm{End}(P)$ [Hint: one way to do this uses part (a)]. The $R$-bilinear map $P^* \times P \to R$ sending $(f, x)$ to $f(x)$ induces an $R$-linear map $\mathrm{Tr} : P^* \otimes P \to R$, and hence a map $\mathrm{Tr} : \mathrm{End}(P) \to R$. Show that if $P = R^n$ is free of rank $n$, then Tr can be identified with the usual trace of matrices in $M_n(R)$.

(c) Suppose now that $\mu : \mathrm{End}(P) \to R$ is any $R$-linear map. Prove that there exists a unique element $g \in \mathrm{End}(P)$ such that $\mu(f) = \mathrm{Tr}(fg)$, for all $f \in \mathrm{End}(P)$.

## Extra Credit Problem

**C1)** (a) If $N$ is a nilpotent $n \times n$ matrix over $\mathbb{C}$, use the binomial series for $\sqrt{1+x}$ to obtain a formula for the square root of $I + N$ (that is, a matrix $A$ with $A^2 = I + N$).

(b) Use the Jordan form to prove that every invertible complex $n \times n$ matrix has a square root.

(c) Suppose $J = J_k(0)$ is a Jordan block of size $k$ for the eigenvalue 0. Find the Jordan canonical form for $J^2$. Use this and part (a) to determine necessary and sufficient conditions for a matrix $A \in M_n(\mathbb{C})$ to have a square root.

(d) Let $F$ be a field of characteristic 2 (such as $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$). Determine necessary and sufficient conditions for a matrix $A \in M_n(F)$ to have a square root (note that you cannot use the idea in part (a) here.)