Math 606 – Fall 2026 – Harry Tamvakis
PROBLEM SET 4 – Due October 1, 2026

1) Consider a polynomial $P \in \mathbb{C}[x, y]$ such that there are no points $(a, b) \in \mathbb{C}^2$ with

$$P(a,b) = \frac{\partial P}{\partial x}(a,b) = \frac{\partial P}{\partial y}(a,b) = 0.$$

Assume also that $V := \{(x,y) \in \mathbb{C}^2 \mid P(x,y) = 0\}$ is connected. Prove that V has the structure of a Riemann surface such that the projections $(x, y) \mapsto x$ and $(x, y) \mapsto y$ are holomorphic functions on V.

2) Let $f \in C_0^\infty(\mathbb{C})$, and let $u \in C^\infty(\mathbb{C})$ be such that $\dfrac{\partial u}{\partial \overline{z}} = f$. Prove that if $\lim\limits_{z \to \infty} |u(z)| = 0$, then

$$u(w) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{f(z)}{z - w}\, dz \wedge d\overline{z}, \quad \forall\, w \in \mathbb{C}.$$

3) Let $u \in C_0^\infty(\mathbb{C})$ and let $n \geq 0$ be an integer. Prove that

$$\iint_{\mathbb{C}} \frac{\partial u}{\partial \overline{z}} \cdot z^n\, dz \wedge d\overline{z} = 0.$$

Prove, conversely, that if $f \in C_0^\infty(\mathbb{C})$ and

$$\iint_{\mathbb{C}} f(z) \cdot z^n\, dz \wedge d\overline{z} = 0, \quad \forall n \geq 0,$$

then

$$u(w) := \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{f(z)}{z - w}\, dz \wedge d\overline{z}$$

has compact support.

4) Let $\phi \in C_0^\infty(\mathbb{C})$ have compact support and $K := \operatorname{supp}(\phi)$. Recall that $\widehat{K}_{\mathbb{C}}$ denotes the union of K with those connected components of $\mathbb{C} \smallsetminus K$ which are relatively compact in $\mathbb{C}$. Prove that if there exists $u \in C_0^\infty(\mathbb{C})$ with $\dfrac{\partial u}{\partial \overline{z}} = \phi$, then the support of u is contained in $\widehat{K}_{\mathbb{C}}$.

5) Construct a connected open set Ω in $\mathbb{C}$ and a function $\phi \in C_0^\infty(\Omega)$ such that the equation $\dfrac{\partial u}{\partial \overline{z}} = \phi$ has a solution $u \in C_0^\infty(\mathbb{C})$ but no solution $u \in C_0^\infty(\Omega)$. Justify your answer.