

Math 648 – Spring 2008 – Harry Tamvakis

PROBLEM SET 4 – Due April 10, 2008

1) Let  $X \rightarrow \mathbb{P}^1$  be the Riemann surface of the algebraic function  $w = \sqrt[n]{1 - z^n}$ , that is, the algebraic function defined by the polynomial  $P(z, w) = w^n + z^n - 1$ . Compute the genus of  $X$ .

2) On the Riemann surface  $X$ , let  $\mathcal{M}$  denote the sheaf of meromorphic functions and let  $\mathcal{N}$  be the sheaf of meromorphic 1-forms which have residue 0 at every point of  $X$ . Prove that the sheaf sequence

$$0 \rightarrow \mathbb{C} \rightarrow \mathcal{M} \xrightarrow{d} \mathcal{N} \rightarrow 0$$

is exact. Deduce from this that  $H^1(X, \mathbb{C}) \cong \mathcal{N}(X)/d\mathcal{M}(X)$ .

3) Recall the operators  $\partial : \mathcal{A}^0 \rightarrow \mathcal{A}^{1,0}$  and  $\bar{\partial} : \mathcal{A}^0 \rightarrow \mathcal{A}^{0,1}$  defined locally by the equations

$$\partial f = \frac{\partial f}{\partial z} dz \quad \text{and} \quad \bar{\partial} f = \frac{\partial f}{\partial \bar{z}} d\bar{z}.$$

Let  $\omega \in A^2(X)$  be a smooth differential 2-form on the compact Riemann surface  $X$ . Prove that there exists a smooth function  $f : X \rightarrow \mathbb{C}$  such that  $\omega = \partial\bar{\partial}f$  if and only if  $\int_X \omega = 0$ .

4) If  $X$  is a compact Riemann surface and  $p \in X$ , prove that  $X \setminus \{p\}$  can be holomorphically embedded in affine space  $\mathbb{C}^n$  for some  $n$ .

5) If  $X$  is a hyperelliptic curve of genus  $g \geq 2$ , prove that the group  $\text{Aut}(X)$  of holomorphic automorphisms  $\phi : X \rightarrow X$  is finite. [Hint: Show that  $\text{Aut}(X)$  permutes the ramification points of the two-fold covering  $X \rightarrow \mathbb{P}^1$ .] One can show that an arbitrary Riemann surface  $X$  has at most  $84(g - 1)$  holomorphic automorphisms, and that this maximum number is achieved for infinitely many values of  $g$ .