Math 660 – Spring 2024 – Harry Tamvakis
PROBLEM SET 4 – Due February 29, 2024

1) Exercises 118 and 119 from the textbook.

2) Let $\Omega \subset \mathbb{C}$ be a domain and $\{f_n\}_{n\geq 1}$ be a sequence of holomorphic functions on Ω. Suppose that there is a function $f : \Omega \to \mathbb{C}$ such that $\lim_{n\to+\infty} f_n(z) = f(z)$ for all $z \in \Omega$. Prove that f is holomorphic on a dense open subset U of Ω, and $f_n \to f$ uniformly on compact subsets of U. [Hint: If V is any relatively compact open subset of Ω and $m \in \mathbb{N}$, let S_m denote the set of all $z \in \overline{V}$ such that $|f_n(z)| \leq m$, for all $n \geq 1$. Use the Baire category theorem to deduce that S_m must contain an open disc D for some $m \geq 1$.]

3) (a) Let Ω be an open set in $\mathbb{C}$, $K \subset \Omega$ a compact subset, and $m \geq 0$ an integer. Prove that there exists a constant C depending on m, K and Ω such that the following is true: For any $f \in \mathcal{O}(\Omega)$, we have

$$\sup_{z\in K} |f^{(m)}(z)| \leq C \iint_{\Omega} |f(z)|\, d\mu,$$

where $f^{(m)}$ is the m-th derivative and $d\mu$ denotes Lebesgue measure.

(b) Show that if $\{f_n\}_{n\geq 1}$ is a sequence in $\mathcal{O}(\Omega)$ and $\{f_n\}$ converges to f in $L^1(\Omega)$, then $f \in \mathcal{O}(\Omega)$ (up to a set of measure zero, which is the equivalence relation in $L^1(\Omega)$).

4) Prove that for every function $f : \Omega \to \mathbb{C}$ holomorphic on an open set $\Omega \subset \mathbb{C}$, and for any two *closed* curves γ_0 and γ_1 mutually homotopic by closed curves γ_u in Ω, we have

$$\int_{\gamma_0} f(z)\, dz = \int_{\gamma_1} f(z)\, dz.$$

(This is true whether or not γ_0 and γ_1 have any point in common.)

5) Exercise 178 from the textbook.