Math 600 – Fall 2025 – Harry Tamvakis
PROBLEM SET 5 – Due October 9, 2025

A1) Let G be a (possibly infinite) group, and suppose that we have subgroups $H \leq K \leq G$. Prove that $|G : H|$ is finite if and only if $|G : K|$ and $|K : H|$ are both finite. In this case, show that $|G : H| = |G : K| \cdot |K : H|$.

A2) Let $\phi : G \to G'$ be a surjective group homomorphism with kernel N. Prove that the set of subgroups H' of G' is in bijective correspondence with the set of subgroups H of G which contain N, under the map which sends H to $\phi(H)$ and H' to $\phi^{-1}(H') := \{g \in G \mid \phi(g) \in H'\}$. Show that the normal subgroups of G' correspond to normal subgroups of G.

A3) (a) Let C_n denote the cyclic group of order n. Prove that the product group $C_m \times C_n$ is cyclic if and only if m and n are relatively prime.

(b) Describe the automorphism group $\mathrm{Aut}(C_n)$ and determine its order. Is $\mathrm{Aut}(C_n)$ a cyclic group for all positive integers n?

B1) (a) A subgroup H of a group G is *proper* if $H \neq G$. A proper subgroup H is *maximal* if it is contained in no other proper subgroup. Find all maximal proper subgroups of the additive group $(\mathbb{Z}, +)$.

(b) A group G is *finitely generated* if there is a finite set $S \subset G$ such that $G = \langle S \rangle$, that is, the subgroup generated by S is all of G. Prove that every proper subgroup of a finitely generated group is contained in a maximal proper subgroup.

(c) Find all maximal proper subgroups of the additive group $(\mathbb{Q}, +)$ of rational numbers, or prove that there are none.

B2) Let G be a (possibly infinite) group.

(a) Suppose that H and K are subgroups of G of finite index in G. Prove that $H \cap K$ also has finite index in G.

(b) Prove that if G has a subgroup $H \neq G$ of finite index, then it has a normal subgroup $N \neq G$ of finite index. It follows that an infinite simple group has no subgroups of finite index.

(c) Refine part (b) by showing that if $|G : H| = k$, then G has a normal subgroup $N \neq G$ with $|G : N| \leq k!$. [Hint: Construct a non-trivial homomorphism from G to the symmetric group S_k.]

B3) A group G is called a *rank one group* if every finitely generated subgroup of G is cyclic. We say that G is *torsion free* if for any $x \in G$ with $x \neq 1$ and any $n \geq 1$, we have $x^n \neq 1$.

(a) Prove that any rank one group is abelian.

(b) Show that the additive group $(\mathbb{Q}, +)$ of rational numbers is a rank one group.

(c) Prove that every torsion-free, rank one group is isomorphic to a subgroup of the group $(\mathbb{Q}, +)$.

B4) Recall that a subset A of $\mathbb{R}^n$ is *bounded* if there exists an $M > 0$ such that $\|a\| \leq M$ for all $a \in A$. Let H be a non-zero subgroup of the additive group $(\mathbb{R}^n, +)$, and assume that the intersection of H with any bounded region in $\mathbb{R}^n$ is a finite set. Prove that H is isomorphic to $(\mathbb{Z}^m, +)$ for some integer m with $1 \leq m \leq n$. [Hint: Use induction on the maximum number of elements of H which are linearly independent vectors in $\mathbb{R}^n$. Do the cases $n = 1$ and $n = 2$ first, before tackling the general case.]

<u>Extra Credit Problems</u>

C1) Prove or give a counterexample: if G is a group such that every proper subgroup of G is finite, then G is finite.

C2) *Klein's* 4-*group* is the unique non-cyclic group of order 4. Prove that a group G is the union of three proper subgroups if and only if there is a surjective group homomorphism from G to Klein's 4-group.