A1) Let $G$ be a (possibly infinite) group, and suppose that we have subgroups $H \leq K \leq G$. Prove that $|G : H|$ is finite if and only if $|G : K|$ and $|K : H|$ are both finite. In this case, show that $|G : H| = |G : K| \cdot |K : H|$. 

A2) Let $\phi : G \to G'$ be a surjective group homomorphism with kernel $N$. Prove that the set of subgroups $H'$ of $G'$ is in bijective correspondence with the set of subgroups $H$ of $G$ which contain $N$, under the map which sends $H$ to $\phi(H)$ and $H'$ to $\phi^{-1}(H') := \{g \in G \mid \phi(g) \in H'\}$. Show that the normal subgroups of $G'$ correspond to normal subgroups of $G$.

A3) Let $C_n$ denote the cyclic group of order $n$. For $m, n \geq 1$, the product group $C_m \times C_n$ is defined as the Cartesian product of $C_m$ with $C_n$ equipped with the pointwise operation $(a, b) \cdot (a', b') := (aa', bb')$. Prove that the group $C_m \times C_n$ is cyclic if and only if $m$ and $n$ are relatively prime.

B1) (a) A subgroup $H$ of a group $G$ is proper if $H \neq G$. A proper subgroup $H$ is maximal if it is contained in no other proper subgroup. Find all maximal proper subgroups of the additive group $(\mathbb{Z}, +)$.

(b) A group $G$ is finitely generated if there is a finite set $S \subset G$ such that $G = \langle S \rangle$, that is, the subgroup generated by $S$ is all of $G$. Prove that every proper subgroup of a finitely generated group is contained in a maximal proper subgroup.

(c) Find all maximal proper subgroups of the additive group $(\mathbb{Q}, +)$ of rational numbers, or prove that there are none.

B2) Let $G$ be a (possibly infinite) group.

(a) Suppose that $H$ and $K$ are subgroups of $G$ of finite index in $G$. Prove that $H \cap K$ also has finite index in $G$.

(b) Prove that if $G$ has a subgroup $H \neq G$ of finite index, then it has a normal subgroup $N \neq G$ of finite index. It follows that an infinite simple group has no subgroups of finite index.

(c) Refine part (b) by showing that if $|G : H| = k$, then $G$ has a normal subgroup $N \neq G$ with $|G : N| \leq k!$. [Hint: Construct a non-trivial homomorphism from $G$ to the symmetric group $S_k$.]

B3) A group $G$ is called a rank one group if every finitely generated subgroup of $G$ is cyclic. We say that $G$ is torsion free if for any $x \in G$ with $x \neq 1$ and any $n \geq 1$, we have $x^n \neq 1$.

(a) Prove that any rank one group is abelian.
(b) Show that the additive group \((\mathbb{Q}, +)\) of rational numbers is a rank one group.

(c) Prove that every torsion-free, rank one group is isomorphic to a subgroup of the group \((\mathbb{Q}, +)\).

**B4)** Recall that a subset \(A\) of \(\mathbb{R}^n\) is *bounded* if there exists an \(M > 0\) such that \(\|a\| \leq M\) for all \(a \in A\). Let \(H\) be a non-zero subgroup of the additive group \((\mathbb{R}^n, +)\), and assume that the intersection of \(H\) with any bounded region in \(\mathbb{R}^n\) is a finite set. Prove that \(H\) is isomorphic to \((\mathbb{Z}^m, +)\) for some integer \(m\) with \(1 \leq m \leq n\).  

[Hint: Use induction on the maximum number of elements of \(H\) which are linearly independent vectors in \(\mathbb{R}^n\). Do the cases \(n = 1\) and \(n = 2\) first, before tackling the general case.]

**Extra Credit Problem**

**C1)** Prove or give a counterexample: if \(G\) is a group such that every proper subgroup of \(G\) is finite, then \(G\) is finite.