Math 601 – Spring 2026 – Harry Tamvakis
PROBLEM SET 5 – Due March 12, 2026

A1) Prove that if an abelian group A is injective as a $\mathbb{Z}$-module, then A is divisible. [Hint: There is a divisible group D with a subgroup isomorphic to A.]

A2) Let R be a commutative ring. Prove that a direct product $\prod_{i \in I} Q_i$ of any family of R-modules Q_i is injective if and only if each Q_i is injective.

A3) Suppose that R is a commutative ring and M and N are flat R-modules. Prove that $M \otimes_R N$ is a flat R-module.

A4) Problem 2 in Chapter 10.5 of Dummit and Foote (page 403).

A5) Consider a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{A}_1 \arrow[r] \arrow[d] & \mathcal{B}_1 \arrow[r] \arrow[d] & \mathcal{C}_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{A}_2 \arrow[r] & \mathcal{B}_2 \arrow[r] & \mathcal{C}_2 \arrow[r] & 0
\end{tikzcd}$$

where each row is an exact sequence of complexes, and each square commutes. Prove the *naturality of the connecting homomorphisms*, i.e., show that for each $n \geq 0$ the following diagram commutes.

$$\begin{tikzcd}
\mathrm{H}^n(\mathcal{C}_1) \arrow[r, "\delta_1"] \arrow[d] & \mathrm{H}^{n+1}(\mathcal{A}_1) \arrow[d] \\
\mathrm{H}^n(\mathcal{C}_2) \arrow[r, "\delta_2"] & \mathrm{H}^{n+1}(\mathcal{A}_2)
\end{tikzcd}$$

B1) Let R be a commutative ring. Suppose that $T : F \to F$ and $T' : F' \to F'$ are endomorphisms of the free R-modules F and F' of ranks m and n, respectively. Then

$$T \otimes T' : F \otimes F' \to F \otimes F'$$

is also an endomorphism of a free R-module.

(a) Show that the trace of $T \otimes T'$ is the product of the traces of T and T'.

(b) Express the determinant of $T \otimes T'$ in terms of the determinants of T and T'.

(c) Do the characteristic polynomials of T and T' determine the characteristic polynomial of $T \otimes T'$?

[NOTE: You may assume that $R = \mathbb{C}$ for this problem. We will discuss later in class a technique which allows one to deduce the more general result above from this special case.]

B2) Let R be a commutative ring.

(a) Suppose $\{M_i\}_{i\in I}$ is a family of R-modules and $f : R \to \bigoplus_{i\in I} M_i$ is a homomorphism of R-modules. Prove that there is a finite subset H of I such that $f(R)$ is contained in $\bigoplus_{i\in H} M_i$.

(b) Suppose that $J_1 \subset J_2 \subset J_3 \subset \cdots$ is an ascending chain of ideals in R and let $J := \bigcup_{i=1}^{\infty} J_i$. For each $i \geq 1$, embed J/J_i into an injective R-module M_i. Assume that $\bigoplus_{i=1}^{\infty} M_i$ is injective. Prove that there is an integer $n \geq 1$ such that the image of J under the natural map

$$J \to \bigoplus_{i=1}^{\infty} J/J_i \to \bigoplus_{i=1}^{\infty} M_i$$

is contained in $\bigoplus_{i=1}^{n} M_i$.

(c) Prove that if R has the property that every countable direct sum of injective R-modules is injective, then R is Noetherian.

B3) Problems 25 and 26 in Chapter 10.5 of Dummit and Foote (page 406).