Math 606 – Fall 2026 – Harry Tamvakis
PROBLEM SET 5 – Due October 8, 2026

1) Suppose that $\mathcal{F}$ is a presheaf of abelian groups on the Riemann surface X and let $|\mathcal{F}| := \coprod_{x\in X}\mathcal{F}_x$ be the associated topological space defined in class, with the natural projection map $p : |\mathcal{F}| \to X$. For each open set $U \subset X$, let $|\mathcal{F}|(U)$ denote the group of all sections of $|\mathcal{F}|$ over U, i.e., the space of all continuous maps $s : U \to |\mathcal{F}|$ such that $p \circ s = \mathrm{id}_U$. Prove the following assertions:

(a) The assignment $U \mapsto |\mathcal{F}|(U)$ together with the natural restriction maps defines a sheaf.

(b) There is a natural isomorphism of the stalks $\mathcal{F}_x \to |\mathcal{F}|_x$ for every $x \in X$.

2) Let X be a Riemann surface and for any open set $U \subset X$, let $\mathcal{F}(U)$ denote the group of all bounded continuous functions $f : U \to \mathbb{R}$. Then $\mathcal{F}$ with the usual restriction maps is a presheaf on X. Show that $\mathcal{F}$ is not a sheaf. Give a description of the associated sheaf $|\mathcal{F}|$.

3) Suppose that $\phi : \mathcal{F} \to \mathcal{G}$ is morphism of sheaves on the Riemann surface X. Prove that ϕ is surjective if and only if the following condition holds: for any open set $U \subset X$ and every section $g \in \mathcal{G}(U)$, there exists an open covering $\{U_i\}_{i\in I}$ of U and elements $f_i \in \mathcal{F}(U_i)$ such that $\phi_{U_i}(f_i) = g|_{U_i}$ for each $i \in I$.

4) A sheaf of abelian groups $\mathcal{F}$ on X is called *flabby* if for every inclusion $V \subset U$ of open sets, the restriction map $r_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ is surjective. Suppose that $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of sheaves on X.

(a) Prove that if $\mathcal{F}$ is flabby, then for every open set U, the sequence

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \to 0$$

is also exact.

(b) Prove that if $\mathcal{F}$ and $\mathcal{G}$ are flabby, then $\mathcal{H}$ is flabby.

5) Let $a_1, \ldots, a_n$ be distinct points in $\mathbb{C}$, and let $\Omega = \mathbb{C} \smallsetminus \{a_1, \ldots, a_n\}$. Let $\mathfrak{U} = \{D_i\}_{i\in I}$ be an open covering of Ω by disks. Prove that $\mathrm{H}^1(\mathfrak{U}, \mathbb{C}) \cong \mathbb{C}^n$. Construct (with proof) an example of a domain Ω' in $\mathbb{C}$ and an open covering $\mathfrak{U}$ of Ω' such that $\mathrm{H}^1(\mathfrak{U}, \mathbb{C})$ is a complex vector space of infinite dimension.