Math 648 – Spring 2008 – Harry Tamvakis
PROBLEM SET 5 – Due April 24, 2008

1) If f and g are two elliptic functions with respect to the same lattice $\Omega \subset \mathbb{C}$, prove that there exists an irreducible polynomial $P(x,y) \in \mathbb{C}[x,y]$ such that $P(f,g)=0$.

2) If f is an elliptic function of order $n>0$ show that f' is elliptic and that its order m satisfies $n+1 \leq m \leq 2n$. Give examples to show that both these bounds are attained.

3) Prove that

$$\wp'(z) = \frac{2\,\sigma(z-\frac{\omega_1}{2})\,\sigma(z-\frac{\omega_2}{2})\,\sigma(z-\frac{\omega_3}{2})}{\sigma(\frac{\omega_1}{2})\,\sigma(\frac{\omega_2}{2})\,\sigma(\frac{\omega_3}{2})\,\sigma(z)^3}\,.$$

4) Prove that for any integer $n \geq 1$, we have

$$\det\left(\wp^{(i+j-1)}(z)\right)_{1\leq i,j\leq n-1} = (-1)^{n-1}\{1!2!\cdots(n-1)!\}\,\frac{\sigma(nz)}{\sigma(z)^{n^2}}\,.$$

Let $\Omega \subset \mathbb{C}$ be a lattice and $\wp(z)$ the associated Weierstrass $\wp$-function. We have seen that $\wp(z)$ satisfies the differential equation $(\wp'(z))^2 = p(\wp(z))$, where $p(x) = 4x^3 - g_2 x - g_3$. The following three problems examine the conditions under which the coefficients g_2 and g_3 of $p(x)$ are real numbers.

5) Prove that the following conditions are equivalent: (i) $g_2, g_3 \in \mathbb{R}$; (ii) $G_k \in \mathbb{R}$ for all $k \geq 3$; (iii) $\wp(\overline{z}) = \overline{\wp(z)}$ for all $z \in \mathbb{C}$; (iv) $\overline{\Omega} = \Omega$ (the last condition says that Ω is a *real lattice*).

6) We say that Ω is *real rectangular* if $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ where $\omega_1 \in \mathbb{R}$ and $\omega_2 \in i\mathbb{R}$, and that Ω is *real rhombic* if $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ where $\omega_2 = \overline{\omega}_1$. Prove that a lattice Ω is real if and only if it is real rectangular or real rhombic.

7) Let Ω be a real lattice. Define the real elliptic curve $E_{\mathbb{R}}$ to be the set $\{(x,y) \in \mathbb{R}^2 \mid y^2 = p(x)\}$. Prove that $E_{\mathbb{R}}$ has one or two connected components as Ω is real rhombic or real rectangular, respectively.