Math 660 – Spring 2020 – Harry Tamvakis

PROBLEM SET 5 – Due March 7, 2024

1) Exercise 116 from the textbook.

2) Exercises 188 and 189 from the textbook.
[Hint: See Proposition 2 on page 72.]

3) It is required to expand the function

$$f(z) = \frac{1}{1-z^2} + \frac{1}{5-z}$$

into a Laurent series of the form $\sum_{n=-\infty}^{+\infty} c_n z^n$. How many such expansions are there, and in which region is each of them valid? Compute the coefficients c_n explicitly for each of these expansions.

4) (a) Let $\lambda \in \mathbb{C}$ and show that

$$\exp\left\{\frac{\lambda}{2}\left(z + \frac{1}{z}\right)\right\} = a_0 + \sum_{n=1}^{\infty} a_n \left(z^n + \frac{1}{z^n}\right)$$

for $0 < |z| < \infty$, where for $n \geq 0$

$$a_n = \frac{1}{\pi}\int_0^{\pi} e^{\lambda \cos t} \cos(nt)\, dt.$$

(b) Similarly, show that

$$\exp\left\{\frac{\lambda}{2}\left(z - \frac{1}{z}\right)\right\} = b_0 + \sum_{n=1}^{\infty} b_n \left(z^n + \frac{1}{z^n}\right)$$

for $0 < |z| < \infty$, where for $n \geq 0$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \cos(\lambda \sin t - nt)\, dt.$$

5) Let f be holomorphic in the punctured disk

$$D^* = \{z \in \mathbb{C} \ : \ 0 < |z| < 1\}$$

and let $d\mu = dxdy$ denote Lebesgue measure in the plane $\mathbb{R}^2$.

(a) Prove that if $\iint_{D^*} |f(z)|^2 \, d\mu < \infty$ then f has a removable singularity at $z = 0$.

(b) Suppose that $p > 0$ and $\iint_{D^*} |f(z)|^p \, d\mu < \infty$. What can be said about the nature of the singularity at $z = 0$?