A1) Let $Z$ be the center of a group $G$. Prove that if $G/Z$ is a cyclic group, then $G$ is abelian and hence $G = Z$.

A2) Let $G$ be any group, finite or infinite. Prove that if $G$ has more than two elements, then Aut$(G)$ $\neq 1$, i.e., $G$ has a non-trivial automorphism.

A3) Let $F$ be any field, and $M_{m,n}(F)$ the vector space of $m \times n$ matrices with entries in $F$. The group $G = GL(m, F) \times GL(n, F)$ acts on $M_{m,n}(F)$ by the rule $(P, Q) \cdot A = PAQ^{-1}$. Describe the decomposition of $M_{m,n}(F)$ into $G$-orbits under this action. Assume that $m \leq n$, and determine the stabilizer of the matrix $(I_m | 0)$.

A4) In each of the following three cases the orthogonal group $G = O(n)$ acts transitively by linear isometries on the given set $X$. Choose a convenient element $x \in X$ and find the stabilizer $G_x$ explicitly as a subgroup of $G$. Conclude in each case that $X$ can be identified with the coset space $G/G_x$.

(a) $X = \{x \in \mathbb{R}^n : \|x\| = 1\}$ is the unit sphere in $\mathbb{R}^n$.

(b) $X$ is the set of one dimensional subspaces of $\mathbb{R}^n$ (i.e. the set of lines through the origin in $\mathbb{R}^n$).

(c) $X$ is the set of $k$-dimensional subspaces of $\mathbb{R}^n$, where $1 \leq k \leq n$.

A5) If $X$ and $Y$ are $G$-sets, we say that a function $\phi : X \to Y$ is a $G$-set homomorphism if $\phi(gx) = g\phi(x)$, for all $g \in G$ and $x \in X$. A bijective homomorphism of $G$-sets is called an isomorphism of $G$-sets. Recall that if $H$ is any subgroup of $G$, then the coset space $G/H$ is a $G$-set under left multiplication by $G$. Prove that if $H$ and $K$ are subgroups of $G$, then the $G$-sets $G/H$ and $G/K$ are isomorphic if and only if $H = gKg^{-1}$ for some $g \in G$.

B1) (a) Let $G$ be a finite group acting on a finite set $X$. For each element $g \in G$, let $X^g = \{x \in X | gx = x\}$ be the subset of elements of $X$ which are fixed by $g$. Prove the formula

$$\sum_{x \in X} |G_x| = \sum_{g \in G} |X^g|.$$  

(b) Suppose that $G$ has exactly $m$ orbits in $X$. Prove that $m|G| = \sum_{g \in G} |X^g|$.

(c) There are $70 = \binom{8}{4}$ ways to color the edges of a regular octagon, making four red and four green. The dihedral group $D_8$ acts on this set of 70 colorings, and the orbits of this action represent equivalent colorings. Determine the number of equivalence classes of such colorings.
B2) Let $G = \text{SL}(2, \mathbb{Z})$ be the group of $2 \times 2$ matrices with integer entries and determinant 1. Pick a prime number $p$ and let $U$ be the set of those $2 \times 2$ integer matrices whose determinant is $p$. $G$ acts on $U$ by left multiplication, i.e., if $\sigma \in G$ and $u \in U$ then $\sigma \cdot u$ is defined to be the matrix product $\sigma u$.

(a) Show that there are exactly $p + 1$ orbits under this action of $G$ on $U$ and that the $p + 1$ matrices

$$w_\infty = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \quad w_r = \begin{pmatrix} 1 & r \\ 0 & p \end{pmatrix}, \quad r = 0, 1, \ldots, p - 1$$

lie one in each orbit. Write $X$ for the set $\{0, 1, \ldots, p - 1, \infty\}$.

(b) If $\sigma \in G$ and $r \in X$, show that there exists a unique $r' \in X$ such that the matrices $w_r \sigma^{-1}$ and $w_r'$ lie in the same orbit. Thus, from $\sigma$ and $r$ we get (unambiguously) an $r'$. Write this relation $P(\sigma)r = r'$. Thus, if $\sigma$ is fixed, $P(\sigma)$ defines a permutation of the set $X$. Show that the map $\sigma \mapsto P(\sigma)$ is a homomorphism $G \to S(X)$. That is, $G$ acts on $X$, via : $\sigma \cdot r = P(\sigma)r$.

(c) Let $N = \text{Ker}P$, where $P$ is the homomorphism above. Prove that $N$ is the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\text{SL}(2, \mathbb{Z})$ such that

$$b, c \equiv 0 \mod p \quad \text{and} \quad a \equiv d \equiv \pm 1 \mod p.$$ 

The group $G/N$ is denoted $\text{PSL}(2, \mathbb{Z}/p\mathbb{Z})$. It is isomorphic to the group if all fractional linear transformations

$$x \mapsto x' = \frac{ax + b}{cx + d}, \quad a, b, c, d \in \mathbb{Z}/p\mathbb{Z}, \quad ad - bc = 1.$$ 

(d) Show that $\text{PSL}(2, \mathbb{Z}/p\mathbb{Z})$ acts transitively on $X$ and that

$$|\text{PSL}(2, \mathbb{Z}/p\mathbb{Z})| = \begin{cases} 6, & \text{if } p = 2 \\ \frac{1}{2}(p^3 - p), & \text{if } p \neq 2. \end{cases}$$

Extra Credit Problem

C1) Prove that a group $G$ is the union of three proper subgroups if and only if there is a surjective group homomorphism from $G$ to Klein’s 4-group.