Math 600 – Fall 2025 – Harry Tamvakis

PROBLEM SET 6 – Due October 16, 2025

A1) Let Z be the center of a group G. Prove that if G/Z is a cyclic group, then G is abelian and hence $G = Z$.

A2) Let F be any field, and $M_{m,n}(F)$ the vector space of $m \times n$ matrices with entries in F. The group $G = \mathrm{GL}(m, F) \times \mathrm{GL}(n, F)$ acts on $M_{m,n}(F)$ by the rule $(P, Q) \cdot A = PAQ^{-1}$. Describe the decomposition of $M_{m,n}(F)$ into G-orbits under this action. Assume that $m \leq n$, and determine the stabilizer of the matrix $(I_m \mid 0)$.

A3) In each of the following three cases the orthogonal group $G = O(n)$ acts transitively by linear isometries on the given set X. Choose a convenient element $x \in X$ and find the stabilizer G_x explicitly as a subgroup of G. Conclude in each case that X can be identified with the coset space G/G_x.

(a) $X = \{x \in \mathbb{R}^n \, : \, \|x\| = 1\}$ is the unit sphere in $\mathbb{R}^n$.

(b) X is the set of one dimensional subspaces of $\mathbb{R}^n$ (i.e. the set of lines through the origin in $\mathbb{R}^n$).

(c) X is the set of k-dimensional subspaces of $\mathbb{R}^n$, where $1 \leq k \leq n$.

A4) If X and Y are G-sets, we say that a function $\phi : X \to Y$ is a *G-set homomorphism* if $\phi(gx) = g\phi(x)$, for all $g \in G$ and $x \in X$. A bijective homomorphism of G-sets is called an *isomorphism* of G-sets. Recall that if H is any subgroup of G, then the coset space G/H is a G-set under left multiplication by G. Prove that if H and K are subgroups of G, then the G-sets G/H and G/K are isomorphic if and only if $H = gKg^{-1}$ for some $g \in G$.

A5) Let $\mathrm{GL}_n(p) := \mathrm{GL}_n(\mathbb{Z}/p\mathbb{Z})$ be the group of invertible $n \times n$ matrices over the field of integers modulo p, for p a prime number.

(a) Compute the order $|\mathrm{GL}_n(p)|$ of the group $\mathrm{GL}_n(p)$.

(b) Find a p-Sylow subgroup of $\mathrm{GL}_n(p)$.

B1) (a) Let G be a finite group acting on a finite set X. For each element $g \in G$, let $X^g = \{x \in X \mid gx = x\}$ be the subset of elements of X which are fixed by g. Prove the formula

$$\sum_{x \in X} |G_x| = \sum_{g \in G} |X^g|.$$

(b) Suppose that G has exactly m orbits in X. Prove that $m|G| = \sum_{g \in G} |X^g|$.

(c) There are $70 = \binom{8}{4}$ ways to color the edges of a regular octagon, making four red and four green. The dihedral group D_8 acts on this set of 70 colorings, and the orbits of this action represent equivalent colorings. Determine the number of equivalence classes of such colorings.

B2) Let $G = \mathrm{SL}(2, \mathbb{Z})$ be the group of 2×2 matrices with integer entries and determinant 1. Pick a prime number p and let U be the set of those 2×2 integer matrices whose determinant is p. G acts on U by left multiplication, i.e., if $\sigma \in G$ and $u \in U$ then $\sigma \cdot u$ is defined to be the matrix product σu.

(a) Show that there are exactly $p+1$ orbits under this action of G on U and that the $p+1$ matrices

$$w_\infty = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \quad w_r = \begin{pmatrix} 1 & r \\ 0 & p \end{pmatrix}, r = 0, 1, \ldots, p-1$$

lie one in each orbit. Write X for the set $\{0, 1, \ldots, p-1, \infty\}$.

(b) If $\sigma \in G$ and $r \in X$, show that there exists a unique $r' \in X$ such that the matrices $w_r \sigma^{-1}$ and $w_{r'}$ lie in the same orbit. Thus, from σ and r we get (unambiguously) an r'. Write this relation $P(\sigma)r = r'$. Thus, if σ is fixed, $P(\sigma)$ defines a permutation of the set X. Show that the map $\sigma \longmapsto P(\sigma)$ is a homomorphism $G \to S(X)$. That is, G acts on X, via : $\sigma \cdot r = P(\sigma)r$.

(c) Let $N = \mathrm{Ker} P$, where P is the homomorphism above. Prove that N is the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{SL}(2, \mathbb{Z})$ such that

$$b, c \equiv 0 \mod p \quad \text{and} \quad a \equiv d \equiv \pm 1 \mod p.$$

The group G/N is denoted $\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z})$. It is isomorphic to the group if all *fractional linear transformations*

$$x \longmapsto x' = \frac{ax+b}{cx+d}, \quad a, b, c, d \in \mathbb{Z}/p\mathbb{Z}, \quad ad - bc = 1.$$

(d) Show that $\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z})$ acts transitively on X and that

$$|\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z})| = \begin{cases} 6, & \text{if } p = 2 \\ \frac{1}{2}(p^3 - p), & \text{if } p \neq 2. \end{cases}$$