Math 601 – Spring 2026 – Harry Tamvakis
PROBLEM SET 6 – Due April 2, 2026

A1) Prove that a finitely generated abelian group A is free if and only if $\mathrm{Ext}^1_{\mathbb{Z}}(A,\mathbb{Z}) = 0$.

A2) Prove that if $0 \to M' \to M \to M'' \to 0$ is a split short exact sequence of R-modules, then for every $n \geq 0$ and R-module N the sequence

$$0 \to \mathrm{Ext}^n_R(M'', N) \to \mathrm{Ext}^n_r(M, N) \to \mathrm{Ext}^n_R(M', N) \to 0$$

is also short exact and split.

A3) Let R be an integral domain and M be a flat R-module. Prove that M is torsion free. Find a natural condition on R and M so that the converse holds.

A4) Let $m, n \geq 1$ be integers. Compute $\mathrm{Tor}^{\mathbb{Z}}_1(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$.

A5) Let F, K, L and M be fields such that $F \subset K \subset M$ and $F \subset L \subset M$, and let KL denote the subfield of M generated by $K \cup L$. Write $[K : F] = m$, $[L : F] = n$ and $[KL : F] = t$ (these cardinalities might be infinite).

(a) Prove that t is finite if and only if both m and n are finite.

(b) If t is finite, show that both m and n divide t, and $t \leq mn$.

(c) If m and n are relatively prime, show that $t = mn$.

A6) If the degree of α over a field F is odd, prove that $F(\alpha) = F(\alpha^2)$.

A7) Find, with proof, the minimal polynomial over $\mathbb{Q}$ of $\sqrt{2} + \sqrt{3}$ and of $\sqrt[3]{2} + \sqrt{3}$.

B1) Let R be a commutative ring.

(a) Let M, M', M'' be R-modules such that there is an exact sequence

$$0 \to M' \to M \to M'' \to 0.$$

If M' and M'' are free R-modules, prove that M is a free R-module.

(b) Let I be an ideal of R. Show that if I is free as an R-module, then I is a principal ideal.

(c) Suppose that R is not a P.I.D. Show that there is an exact sequence as in part (a) where M is free but neither M' nor M'' is free.

B2) Let $0 \to M' \to M \to M'' \to 0$, $0 \to A' \to P' \to M' \to 0$, and $0 \to A'' \to P'' \to M'' \to 0$ be exact sequences of modules over a commutative ring. Assume that P'' is projective. Show that these sequences may be put into a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P' \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & P'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

in which the rows and columns are exact and the middle row splits.

B3) (a) Let R and S be commutative rings and $f : R \to S$ a ring homomorphism making S into an R-module. Prove that if S is flat as an R-module, then $\mathrm{Tor}_n^R(M, N) \cong \mathrm{Tor}_n^S(S \otimes_R M, N)$ for all R-modules M and S-modules N. [Hint: Show that tensoring an R-module projective resolution for M with S gives an S-module projective resolution of $S \otimes_R M$.]

(b) Let $D^{-1}R$ be the localization of the commutative ring R with respect to the multiplicative subset D of R containing 1. Prove that $D^{-1}R$ is flat over R, or equivalently, that localization of modules is an exact functor.

(c) Prove that localization commutes with Tor, i.e.,

$$D^{-1}\mathrm{Tor}_n^R(M, N) \cong \mathrm{Tor}_n^{D^{-1}R}(D^{-1}M, D^{-1}N)$$

for all R-modules M and N and all $n \geq 0$.

(d) Given any R-module M and prime ideal P of R, let R_P (resp. M_P) denote the localization of R (resp. M) with respect to $D = R \setminus P$. Prove that an R-module M is flat if and only if M_P is a flat for every maximal (hence also for every prime) ideal P in R.

Extra Credit Problem.

C1) Show that there does not exist a polynomial $f(x) \in \mathbb{Z}[x]$ of degree greater than 1 that is irreducible modulo p for all prime numbers p.