A1) Let $K$ and $L$ be subfields of a field $M$ and suppose that $K$ and $L$ both contain the field $F$. Let $KL$ denote the subfield of $M$ generated by $K \cup L$. Write $[K : F] = m$, $[L : F] = n$ and $[KL : F] = t$ (these cardinalities might be infinite).

(a) Prove that $t$ is finite if and only if both $m$ and $n$ are finite.
(b) In this case show that both $m$ and $n$ divide $t$, and $t \leq mn$.
(c) If $m$ and $n$ are relatively prime, show that $t = mn$.

A2) If the degree of $\alpha$ over a field $F$ is odd, prove that $F(\alpha) = F(\alpha^2)$.

A3) Let $\alpha$ be a complex root of the polynomial $x^3 - 3x + 4$, which is irreducible over $\mathbb{Q}$. Find the inverse of $\alpha^2 + \alpha + 1$ in $\mathbb{Q}(\alpha)$ explicitly, in the form $u + v\alpha + w\alpha^2$, where $u, v, w \in \mathbb{Q}$.

A4) Suppose that $\alpha$ and $\beta$ have minimal polynomials $x^2 + a_1x + a_2$ and $x^2 + b_1x + b_2$ over a field $F$, respectively.

(a) Use the method explained in class (or your own approach) to construct a polynomial in $F[x]$ which has $\alpha \beta$ as a root.
(b) Which polynomial does part (a) produce when $F = \mathbb{Q}$, $\alpha = \sqrt{m}$, and $\beta = \sqrt{n}$, where $m$ and $n$ are positive integers which are not perfect squares?

A5) Find, with proof, the minimal polynomial over $\mathbb{Q}$ of $\sqrt{2} + \sqrt{3} + \sqrt{5}$.

A6) Prove that if $m$ is an integer which is not a perfect square and if $a + b\sqrt{m}$ with $a, b \in \mathbb{Q}$ is the root of a polynomial $p(x) \in \mathbb{Q}[x]$, then $a - b\sqrt{m}$ is also a root of $p(x)$.

A7) In class we explained that a real number $a$ is called constructible if, given the origin $(0, 0)$ and the point $(1, 0)$, we can use a straight edge and compass to construct the point $(a, 0)$.

(a) Prove that the set $K$ of constructible real numbers is a subfield of $\mathbb{R}$.
(b) Prove that the field $K$ is the smallest subfield of $\mathbb{R}$ with the property that if $a > 0$ and $a \in K$, then $\sqrt{a} \in K$.

B1) (a) Let $R$ and $S$ be commutative rings and $f : R \to S$ a ring homomorphism making $S$ into an $R$-module. Prove that if $S$ is flat as an $R$-module, then $\text{Tor}_n^R(M, N) \cong \text{Tor}_n^S(S \otimes_R M, N)$ for all $R$-modules $M$ and $S$-modules $N$. [Hint: Show that tensoring an $R$-module projective resolution for $M$ with $S$ gives an $S$-module projective resolution of $S \otimes_R M$.]
(b) Let $D^{-1}R$ be the localization of the commutative ring $R$ with respect to the multiplicative subset $D$ of $R$ containing 1. Prove that $D^{-1}R$ is flat over $R$, or equivalently, that localization of modules is an exact functor.

(c) Prove that localization commutes with Tor, i.e.,
$$D^{-1}\text{Tor}_n^R(M, N) \cong \text{Tor}_n^{D^{-1}R}(D^{-1}M, D^{-1}N)$$
for all $R$-modules $M$ and $N$ and all $n \geq 0$.

(d) Given any $R$-module $M$ and prime ideal $P$ of $R$, let $R_P$ (resp. $M_P$) denote the localization of $R$ (resp. $M$) with respect to $D = R \setminus P$. Prove that an $R$-module $M$ is flat if and only if $M_P$ is a flat for every maximal (hence also for every prime) ideal $P$ in $R$.

**B2)** Let $F$ be any field.

(a) Suppose that the additive group $(F, +)$ is a finitely generated abelian group. Prove that $F$ must be a finite field.

(b) Suppose that $F$ is a finite field. Classify the additive subgroup $(F, +)$ up to group isomorphism.

**B3)** Let $n \geq 1$ be positive integer and
$$\zeta_n := e^{2\pi i/n} = \cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right).$$
Find the minimal polynomial over $\mathbb{Q}$ of (a) $\zeta_6$ (b) $\zeta_9$ (c) $\zeta_{11}$ (d) $\zeta_{12}$.

**C problem**

(a) Prove that it is possible to divide a $19^\circ$ angle into 19 equal parts with a straight edge and compass.

(b) In part (a) you constructed a $1^\circ$ angle. Why doesn’t this contradict the discussion in class on trisecting an angle of $60^\circ$?