Math 606 – Fall 2026 – Harry Tamvakis
PROBLEM SET 6 – Due October 15, 2026

1) Suppose that X is a Riemann surface and s is a (not identically zero) meromorphic section of a holomorphic vector bundle $\pi : E \to X$. For $a \in X$ choose a coordinate chart (U, z) for X at a with $z(a) = 0$ and a local trivialization $\phi : \pi^{-1}(E) \to U \times \mathbb{C}^n$ such that $\phi(s(x)) = (x, f(x))$ for $x \in U$, where $f = (f_1, \ldots, f_n)$ is an n-tuple of meromorphic functions on U. We defined $\mathrm{ord}_a(s)$ to be the integer k such that $f = z^k g$ where $g = (g_1, \ldots, g_n)$ with each g_i holomorphic near a and $g(a) \neq 0$. Prove that k is uniquely determined and does not depend on the choice of coordinate chart and local trivialization.

2) Let E, E' be holomorphic vector bundles of ranks n and m over the Riemann surface X, with transition functions $\{g_{ij}\}$ and $\{g'_{ij}\}$ with respect to the same open cover $\{U_i\}_{i \in I}$ of X. Let $M_{m,n}(\mathbb{C})$ denote the vector space of all $m \times n$ matrices with entries in $\mathbb{C}$.

(a) Show that a morphism of vector bundles $E \to E'$ may be identified with a family of holomorphic maps $\{f_i : U_i \to M_{m,n}(\mathbb{C})\}_{i \in I}$ such that for each $i, j \in I$ we have $f_i g_{ij} = g'_{ij} f_j$ on $U_i \cap U_j$.

(b) Let $\mathrm{Pic}(X)$ denote the group of isomorphism classes of holomorphic line bundles on X with the group structure given by the tensor product of line bundles. Show that $\mathrm{Pic}(X)$ is isomorphic to the cohomology group $H^1(X, \mathcal{O}^*)$.

3) Let L be a holomorphic line bundle on a compact Riemann surface X. Prove that L is trivial if and only if both L and its dual L^* have non-trivial global holomorphic sections.

4) Let X be a Riemann surface and $p : E \to X$ be a holomorphic vector bundle of rank n on X. A *holomorphic subbundle* $F \subset E$ of rank k is a subset such that the following holds. For every $x \in X$, there exists a local trivialization $\phi : p^{-1}(U) \to U \times \mathbb{C}^n$ of E with $x \in U$ such that $\phi(p^{-1}(U) \cap F) = U \times (\mathbb{C}^k \times 0)$.

(a) Let $s : X \to E$ be a holomorphic section of E which is nowhere zero. For $x \in X$, define $L_x := \mathbb{C} \cdot s(x) \subset E_x$. Prove that

$$L := \bigcup_{x \in X} L_x \subset E$$

is a holomorphic subbundle of E of rank 1.

(b) Let s be a meromorphic section of E over X. Show that there exists a unique subbundle $L \subset E$ of rank 1 such that s is a meromorphic section of L.

5) If $D = \sum_P n_P[P]$ is any divisor on a compact Riemann surface X, the *degree* of D is the integer $\deg(D) := \sum_P n_P$. If $L \to X$ is a holomorphic line bundle, then the degree of L, denoted $\deg(L)$, is defined as the degree of the divisor of a nonzero meromorphic section s of L over X. We will show in class that such a section always exists.

(a) Show that $\deg(L)$ is well defined, that is, it does not depend on the choice of meromorphic section s.

(b) On the Riemann sphere $\mathbb{P}^1$, let $\mathfrak{U} := \{U_1, U_2\}$ be the open covering given by

$$U_1 := \{z \in \mathbb{C} \ : \ |z| < 1+\epsilon\},\ U_2 := \{z \in \mathbb{P}^1 \ : \ |z| > 1-\epsilon\}, \ \ 0 < \epsilon < 1.$$

For any integer k, let L_k denote the holomorphic line bundle on $\mathbb{P}^1$ given by the transition function $g_{12} : U_1 \cap U_2 \to \mathbb{C}^*$ with $g_{12}(z) := z^k$. Prove that $\deg(L_k) = k$.

(c) Suppose that L is the holomorphic line bundle on $\mathbb{P}^1$ given by an arbitrary transition function $g_{12} : U_1 \cap U_2 \to \mathbb{C}^*$. Prove that

$$\deg(L) = \frac{1}{2\pi i} \int_{|z|=1} \frac{g_{12}'(z)}{g_{12}(z)}\, dz.$$