Math 660 – Spring 2024 – Harry Tamvakis
PROBLEM SET 6 – Due March 28, 2024

1) Prove the identity

$$\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^{n+1}} = \frac{\pi}{4^n}\frac{(2n)!}{(n!)^2}$$

for n any nonnegative integer.

2) Prove the identity

$$\int_{-\infty}^{+\infty} \frac{dx}{(x^4+a^4)^2} = \frac{3}{8}\frac{\sqrt{2}}{a^7}\,\pi$$

for a any positive real number.

3) Let $u \in C_0^\infty(\mathbb{C})$ and let $n \geq 0$ be an integer. Prove that

$$\iint_{\mathbb{C}} \frac{\partial u}{\partial \overline{z}} \cdot z^n \, dz \wedge d\overline{z} = 0.$$

Prove, conversely, that if $f \in C_0^\infty(\mathbb{C})$ and

$$\iint_{\mathbb{C}} f(z) \cdot z^n \, dz \wedge d\overline{z} = 0, \quad \forall n \geq 0,$$

then

$$u(w) = \frac{1}{2\pi i}\iint_{\mathbb{C}} \frac{f(z)}{z-w}\, dz \wedge d\overline{z}$$

has compact support.

Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$, so that $\Lambda := \mathbb{Z}\,\omega_1 + \mathbb{Z}\,\omega_2$ is a lattice in $\mathbb{C}$. An *elliptic function* is a meromorphic function f on $\mathbb{C}$ such that $f(z+\omega) = f(z)$ for all $\omega \in \Lambda$ and every $z \in \mathbb{C}$ where f is holomorphic. For $z_0 \in \mathbb{C}$ let

$$P(z_0) := \{z_0 + t_1\omega_1 + t_2\omega_2 \mid 0 \leq t_1 \leq 1,\, 0 \leq t_2 \leq 1\}$$

and assume that the boundary $\partial P(z_0)$ does not contain any pole of f.

4) (a) Prove that $\displaystyle\int_{\partial P(z_0)} f(z)\, dz = 0.$

(b) If $\partial P(z_0)$ does not contain any zero of f, prove that the interior of $P(z_0)$ contains equally many zeroes as poles of f, all counted with their multiplicities.

5) Assume that $\partial P(z_0)$ does not contain any zero or pole of f, and let $z_1, \ldots, z_n$ (resp. $w_1, \ldots, w_n$) be the zeroes (resp. poles) of f in the interior of $P(z_0)$, counted with their multiplicities. Prove that

$$\sum_{i=1}^{n} (z_i - w_i) \in \Lambda.$$

[Hint: Consider the integral $\displaystyle\int_{\partial P(z_0)} \frac{z f'(z)}{f(z)} \, dz$.]