Math 406 – Fall 2025 – Harry Tamvakis
PROBLEM SET 7 – Due October 30, 2025

Reading for this week: Section 9.

Problems

From the textbook: Section 9, Problems #2, 4, 5, 6, 14, 16, 18, 19, 20. In addition, do the following problems:

A1) Find the last two digits in the decimal expansions of 3^{1000} and 7^{999999}.

A2) Prove that $2^{15} - 2^3$ divides $a^{15} - a^3$ for any integer a. [Hint: $2^{15} - 2^3 = 5 \cdot 7 \cdot 8 \cdot 9 \cdot 13$.]

A3) A deck of cards is shuffled by cutting the deck into two piles of 26 cards. Then, the new deck is formed by alternating cards from the two piles, starting with the bottom pile.

(a) Show that if a card begins in the k-th position in the deck, it will be in the d-th position in the new deck, where $d \equiv 2k \pmod{53}$ and $1 \leq d \leq 52$.

(b) Determine the number of shuffles of the type described above that are needed to return the deck of cards to its original order.

Extra Credit Problems.

EC1) (a) Consider the number $m = 111 \cdots 1$ with n digits, all ones. Prove that if m is prime, then n is prime.

(b) Is the converse of the statement in (a) true?

EC2) Consider the canonical factorization of 1000! into prime powers:

$$1000! = 2^a 3^b 5^c 7^d 11^e \cdots$$

Compute the exponents a, b, c, and d.