Math 600 – Fall 2025 – Harry Tamvakis
PROBLEM SET 7 – Due October 23, 2025

A1) Let G be a finite group. Suppose that G has a unique p-Sylow subgroup for each prime p dividing $|G|$. Prove that G is the direct product of its Sylow subgroups.

A2) Let G be a finite group, let P be a p-Sylow subgroup of G and H be any subgroup. Show that if H contains the normalizer $N_G(P)$, then $N_G(H) = H$. In particular, $N_G(N_G(P)) = N_G(P)$.

A3) Let G be the group of rotational symmetries of a regular tetrahedron. Describe the stabilizer of a vertex and the stabilizer of an edge. Find $|G|$. Prove that G is isomorphic to the alternating group A_4.

A4) Suppose $w \in S_n$ and let k_i denote the number of i-cycles in the decomposition of w as a product of disjoint cycles, for $1 \leq i \leq n$. Show that the conjugacy class of w has $n! \Big/ [(\prod i^{k_i})(\prod k_i!)]$ elements.

B1) Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and S_∞ be the set of bijections $\sigma : \mathbb{N} \to \mathbb{N}$ such that there is a number M (depending on σ) so that $\sigma(n) = n$ if $n \geq M$.

(a) Prove that S_∞ is a subgroup of $S(\mathbb{N}) = \mathrm{Perm}(\mathbb{N})$.

(b) Show that the notion of an even permutation makes sense in S_∞. Let A_∞ be the subgroup of even permutations in S_∞. Prove that A_∞ has index 2 in S_∞, and therefore is a normal subgroup of S_∞.

(c) It is known that A_n is a *simple* group provided $n \geq 5$. Assume this and prove that A_∞ is also simple.

(d) Show that S_∞ has only two subgroups of finite index. However, prove that every finite group is isomorphic to a subgroup of S_∞.

(e) Prove that neither the additive group $(\mathbb{Z}, +)$ of integers nor the group $\mathrm{GL}(2, \mathbb{Z})$ of invertible 2×2 matrices with integer entries are isomorphic to subgroups of S_∞.

(f) Finally, prove that if H is a finitely generated subgroup of S_∞, then H is a finite group.

B2) (a) For any permutation $w \in S_n$, let $r(w)$ denote the number of pairs (i, j) with $1 \leq i < j \leq n$ such that $w(i) > w(j)$:

$$r(w) := \#\{(i, j) \mid i < j \ , \ w(i) > w(j)\}.$$

What is the set of all possible values $r(w)$ as w ranges over permutations in S_n? Justify your answer.

(b) For each i with $1 \leq i \leq n-1$, let $s_i = (i, i+1)$ denote the transposition which interchanges i and $i+1$ in the symmetric group S_n. Show that the set $\{s_1, \ldots, s_{n-1}\}$ generates all of S_n.

(c) For each pair of integers i and j with $1 \leq i \leq j \leq n-1$, find the least positive integer m_{ij} such that $(s_i s_j)^{m_{ij}} = 1$. Determine the matrix $\{m_{ij}\}$ when $n = 5$.

(d) For any $w \in S_n$, the *length* of w, denoted $\ell(w)$, is the minimal length k of a sequence $a_1, \ldots, a_k$ such that $w = s_{a_1} \cdots s_{a_k}$. Prove that $\ell(w) = r(w)$.

(e) For $i < j$, let t_{ij} denote the transposition (or 2-cycle) (ij) which interchanges i and j. Find a simple criterion in terms of w, i, and j that ensures that $\ell(wt_{ij}) = \ell(w) + 1$.

B3) (a) Let G be a group and $S \subset G$ be a generating set such that $1 \notin S$. The *Cayley graph* $C(G, S)$ of the pair (G, S) has a vertex for every element g in G and an edge connecting $g \in G$ to $h \in G$ if and only if there is an $s \in S \cup S^{-1}$ such that $h = gs$.

We'll say that S as above is a *minimal generating set* if no proper subset of S generates G. In each case below choose a minimal generating set S for the given group G and draw the Cayley graph $C(G, S)$:

$$\text{(i)}\, C_3 \times C_3\,; \quad \text{(ii)}\, C_2 \times C_2 \times C_2\,; \quad \text{(iii)}\, D_5\,;$$
$$\text{(iv)}\, Q \text{ (the quaternion group)}\,; \quad \text{(v)}\, (\mathbb{Z}^2, +).$$

(b) Let $G := S_4$ and $S := \{(12), (23), (34)\} = \{s_1, s_2, s_3\}$, as in problem B2 above. Draw a picture of the Cayley graph $C(G, S)$, positioning the vertices in rows according to the length of each permutation.

<u>Extra Credit Problem</u>

C1) Let t be a formal variable and $\ell : S_n \to [0, +\infty)$ be the length function defined in problem B2. Prove the identity

$$\sum_{w \in S_n} t^{\ell(w)} = \prod_{k=1}^{n} \frac{t^k - 1}{t - 1} = (1+t)(1+t+t^2) \cdots (1+t+\cdots+t^{n-1}).$$