

**Math 601 – Spring 2026 – Harry Tamvakis**  
**PROBLEM SET 7 – Due April 9, 2026**

**A1)** Let  $n \geq 1$  be positive integer and

$$\zeta_n := e^{2\pi i/n} = \cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right).$$

Find the minimal polynomial over  $\mathbb{Q}$  of (a)  $\zeta_5$  (b)  $\zeta_6$  (c)  $\zeta_8$  (d)  $\zeta_{10}$ .

**A2)** Let  $K_1$  and  $K_2$  be two finite extensions of a field  $F$  which are contained in the field  $L$ . Assume that both  $K_1$  and  $K_2$  are splitting fields (of some polynomials) over  $F$ . Prove that the composite field  $K_1K_2$  and the intersection  $K_1 \cap K_2$  are both splitting fields over  $F$ .

**A3)** Let  $K_1$  and  $K_2$  be two finite extensions of a field  $F$  which are both contained in the field  $L$ . Prove that the  $F$ -algebra  $K_1 \otimes_F K_2$  is a field if and only if  $[K_1K_2 : F] = [K_1 : F][K_2 : F]$ .

**A4)** Show that if  $F$  is an infinite field, then the multiplicative group  $(F^*, \cdot)$  is not cyclic.

**A5)** (a) Let  $p$  be a prime and  $f(x)$  be an irreducible polynomial of degree  $n$  over the finite field  $\mathbb{F}_p$ . Show that  $\mathbb{F}_p[x]/(f(x))$  is a field with  $p^n$  elements, which is isomorphic to a splitting field of the polynomial  $x^{p^n} - x$  and  $f(x)$  divides  $x^{p^n} - x$ .

(b) Let  $g(x)$  be an irreducible polynomial in  $\mathbb{F}_p[x]$ . Show that  $g(x)$  divides  $x^{p^n} - x$  if and only if  $\deg(g(x))$  divides  $n$ .

**A6)** (a) Prove that  $x^3 + x^2 + 1$  and  $x^3 + x + 1$  are irreducible over the finite field  $\mathbb{F}_2$ .

(b) Give an explicit isomorphism between the quotients  $\mathbb{F}_2[x]/(x^3 + x^2 + 1)$  and  $\mathbb{F}_2[x]/(x^3 + x + 1)$ .

**A7)** For any prime  $p$  and non-zero  $a \in \mathbb{F}_p$ , prove that the polynomial  $x^p - x + a$  is irreducible and separable over  $\mathbb{F}_p$ . [Hint: Prove that if  $\alpha$  is root then  $\alpha + 1$  is also a root.]

**B1)** A complex number  $z$  is called an *algebraic integer* if  $z$  is a root of a monic polynomial  $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$  with integer coefficients  $a_i$ , for some  $n \geq 1$ .

(a) Prove that if  $\alpha_1, \dots, \alpha_k$  are algebraic integers, then  $\mathbb{Z}[\alpha_1, \dots, \alpha_k]$  is a finitely generated abelian group.

(b) Prove that the set  $\mathcal{O}$  of all algebraic integers is a *subring* of  $\mathbb{C}$ .

(c) Suppose that  $z$  is a root of a monic polynomial  $g(x) = x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_1x + \alpha_0$  with coefficients  $\alpha_i \in \mathcal{O}$ . Prove that  $z \in \mathcal{O}$ .

**B2)** Let  $F$  be any field.

(a) Suppose that the additive group  $(F, +)$  is a finitely generated abelian group. Prove that  $F$  must be a finite field.

(b) Suppose that  $F$  is a finite field. Classify the additive subgroup  $(F, +)$  up to group isomorphism.

**B3)** Let  $F$  be a field and  $F(x)$  be the quotient field of the polynomial ring  $F[x]$ . Let  $t \in F(x)$  be the rational function  $t = p(x)/q(x)$ , where  $p(x), q(x) \in F[x]$  are relatively prime polynomials and  $q(x) \neq 0$ . Then  $F(x)$  is an extension of the field  $F(t)$ .

(a) Show that the polynomial  $p(z) - tq(z)$  in the variable  $z$  and coefficients in  $F(t)$  is irreducible over  $F(t)$  and has  $x$  as a root. [Hint: By Gauss' Lemma this polynomial is irreducible in  $(F(t))[z]$  if and only if it is irreducible in  $(F[t])[z]$ . Then note that  $(F[t])[z] = (F[z])[t]$ .]

(b) Show that  $[F(x) : F(t)] = \max(\deg p(x), \deg q(x))$ .

(c) Show that  $F(t) = F(x)$  if and only if

$$t(x) = \frac{ax + b}{cx + d}$$

for some  $a, b, c, d \in F$  such that  $ad - bc \neq 0$ .

### C problem

**C1)** Let  $\phi$  denote the Frobenius map  $x \mapsto x^p$  on the finite field  $\mathbb{F}_{p^n}$ . Consider  $\phi$  as an  $\mathbb{F}_p$ -linear transformation of the  $n$ -dimensional  $\mathbb{F}_p$ -vector space  $\mathbb{F}_{p^n}$ .

(a) Determine the rational canonical form over  $\mathbb{F}_p$  for  $\phi$ .

(b) Determine the Jordan canonical form for  $\phi$  (over a field which contains all the eigenvalues of  $\phi$ ).