Math 606 – Fall 2026 – Harry Tamvakis
PROBLEM SET 7 – Due October 29, 2026

1) Let X be a compact Riemann surface and let L be a holomorphic line bundle on X. Let $\{g_{ij}\}$ be the transition functions for L (with respect to some open cover of X) and let L^k be the line bundle with transition functions $\{g_{ij}^k\}$. By analyzing the proof of the finiteness theorem, prove the following: There exists a constant $C > 0$ depending only on L (and the $\{g_{ij}\}$) such that

$$\dim \mathrm{H}^0(X, L^k) \leq C \cdot k \quad \text{for} \quad k \geq 1.$$

2) Let V be a *topological vector space*, that is, a topological space which is also a complex vector space such that (i) every point in V is a closed set and (ii) the vector space operations are continuous. Prove that the topology on V is Hausdorff.

3) Let D be a divisor on the Riemann sphere $\mathbb{P}^1$. Prove that

(a) $\dim \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_D) = \max(0, 1 + \deg D)$

(b) $\dim \mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_D) = \max(0, -1 - \deg D)$.

4) Let Ω be a lattice in $\mathbb{C}$ and $X = \mathbb{C}/\Omega$ be the corresponding torus, P be a point on X, and k be an integer. Prove that

$$\dim \mathrm{H}^0(X, \mathcal{O}_{kP}) = \begin{cases} 0 & \text{if } k < 0, \\ 1 & \text{if } k = 0, \\ k & \text{if } k \geq 1. \end{cases}$$

[Hint: Use the Weierstrass $\wp$-function.]

5) Let X be a compact Riemann surface, D a divisor on X, and $\mathfrak{U} = \{U_i\}$ be an opening covering of X such that every U_i is biholomorphic to a disk in $\mathbb{C}$. Prove that $\mathrm{H}^1(X, \mathcal{O}_D) = \mathrm{H}^1(\mathfrak{U}, \mathcal{O}_D)$.