Math 660 – Spring 2024 – Harry Tamvakis
PROBLEM SET 7 – Due April 4, 2024

1) Let Ω be a *bounded* connected open set in $\mathbb{C}$ and $a \in \Omega$. Let $f : \Omega \to \Omega$ be a holomorphic map such that $f(a) = a$.

(a) Define the iterates of f by $f_1 := f$ and $f_{k+1} := f \circ f_k$ for $k \geq 1$. Compute $f_k'(a)$ and deduce that $|f'(a)| \leq 1$.

(b) If $f'(a) = 1$, prove that $f(z) = z$ for all $z \in \Omega$. [Hint: If

$$f(z) = z + c_m(z-a)^m + \cdots ,$$

compute the coefficient of $(z-a)^m$ in the expansion of $f_k(z)$.]

(c) If $|f'(a)| = 1$, prove that f is injective and that $f(\Omega) = \Omega$. [Hint: If $\rho := f'(a)$, there is an increasing sequence $n_1 < n_2 < \cdots$ of positive integers n_k such that $\rho^{n_k} \to 1$ and $f_{n_k} \to g$ as $k \to \infty$. Show that $g'(a) = 1$ and $g(\Omega) \subset \Omega$, hence $g(z) = z$ by part (b). Use g to draw the desired conclusions about f.]

2) Exercise 248 in the textbook.

3) Exercise 249 in the textbook.

4) Exercise 257 in the textbook.

5) Exercise 258 in the textbook.