A1) Let $G$ be a finite group. Suppose that $G$ has a unique $p$-Sylow subgroup for each prime $p$ dividing $|G|$. Prove that $G$ is the direct product of its Sylow subgroups.

A2) Let $G$ be a finite group, let $P$ be a $p$-Sylow subgroup of $G$ and $H$ be any subgroup. Show that if $H$ contains the normalizer $N_G(P)$, then $N_G(H) = H$. In particular, $N_G(N_G(P)) = N_G(P)$.

A3) If $G$ is a group of order 231, prove that the 11-Sylow subgroup is in the center of $G$.

A4) (a) Let $p$ be an odd prime and let $G$ be a group of order $2^p$. Prove that the set $G^2 := \{g^2 \mid g \in G\}$ is a subgroup of $G$.

      (b) Let $A_4$ be the group of even permutations of four objects. Show that the set $A_4^2 := \{\sigma^2 \mid \sigma \in A_4\}$ is not a subgroup of $A_4$.

A5) Let $G = N \rtimes H$, and suppose that $K$ is a subgroup of $G$ with $N \subset K$. Prove that $K = N \rtimes (K \cap H)$.

A6) Let $F$ be a field and let $G$ be the group of upper triangular matrices in $GL(n, F)$.

      (a) Prove that $G = U \rtimes D$, where $U$ is the subgroup of $G$ consisting of matrices with with 1’s along the diagonal, and $D$ is the group of diagonal matrices in $GL(n, F)$.

      (b) Suppose that $n = 2$. In this case note that $U \cong (F, +)$ and $D \cong (F^* \times F^*, \cdot)$. Describe the conjugation homomorphism $D \to \text{Aut}(U)$ explicitly in terms of these isomorphisms (i.e., show how each element of $F^* \times F^*$ acts as an automorphism on $F$).

B1) (a) Suppose that $H$ is a subgroup of a group $G$ of index 2. If $K$ is a subgroup of $G$ of odd order, prove that $K \subset H$.

      (b) Let $G$ be a finite group and suppose that there exist subgroups

      \[ G = G_0 \supset G_1 \supset \cdots \supset G_r = H \]

      with $|G_i : G_{i+1}| = 2$ for all $i$ with $0 \leq i < r$. If $|H|$ is odd, prove that $H$ is normal in $G$.

      (c) Let $G$ be a group of order $2^k m$, where $m$ is odd. Suppose that $G$ contains a normal subgroup $H$ of order $m$. Prove that there exist subgroups

      \[ G = G_0 \supset G_1 \supset \cdots \supset G_r = H \]
with $|G_i : G_{i+1}| = 2$ for all $i$ with $0 \leq i < r$.

**B2** Let $N$ and $H$ be groups. An extension of $N$ by $H$ is a group $E$ along with a monomorphism $\phi : N \to E$ and an epimorphism $\psi : E \to H$ such that $\phi(N) = \text{Ker}(\psi)$. In this case $N$ embeds in $E$ as a normal subgroup, with the quotient group being isomorphic to $H$. We identify $N$ with its image under $\phi$, and $H$ with the quotient group $E/N$.

(a) We say that an extension $E$ of $N$ by $H$ is a split extension if there is a homomorphism $\epsilon : H \to E$ (called the splitting map for the extension) such that $\psi \circ \epsilon$ is the identity map on $H$. Show that $E$ is a split extension of $N$ by $H$ if and only if $E$ is a semidirect product of $N$ and $H$.

(b) Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ denote the quaternion group of order 8. Show that $Q_8$ can be realized as a non-trivial extension in four ways – thrice as an extension of $C_4$ by $C_2$, and once as an extension of $C_2$ by $C_2 \times C_2$, but that none of these extensions is split. This proves that $Q_8$ cannot be written in a non-trivial way as a semidirect product.

**Extra Credit Problems**

**C1** Prove that all groups of order less than 60 are solvable. [Hint: Show that if $|G| = n < 60$, then for some prime $p$ dividing $n$, the number $m$ of $p$-Sylow subgroups of $G$ does not exceed 4. If $m > 1$, consider the action of $G$ by conjugation on the set of all $p$-Sylow subgroups and obtain a non-trivial homomorphism $G \to S_m$.]

**C2** Follow up on problem B2 of homework #6 and prove: if $p$ is a prime with $p \geq 5$, then $\text{PSL}(2, \mathbb{Z}/p\mathbb{Z})$ is a simple group. (This is the second infinite family of non-abelian finite simple groups; the alternating groups $A_n$ for $n \geq 5$ were the first found. There is one overlap: $A_5 \cong \text{PSL}(2, \mathbb{Z}/5\mathbb{Z})$).