Math 600 – Fall 2025 – Harry Tamvakis
PROBLEM SET 8 – Due November 6, 2025

A1) If G is a group of order 231, prove that the 11-Sylow subgroup is in the center of G.

A2) (a) Let p be an odd prime and let G be a group of order $2p$. Prove that the set $G^2 := \{g^2 \mid g \in G\}$ is a subgroup of G.

(b) Let A_4 be the group of even permutations of four objects. Show that the set $A_4^2 := \{\sigma^2 \mid \sigma \in A_4\}$ is not a subgroup of A_4.

A3) Let $G = N \rtimes H$, and suppose that K is a subgroup of G with $N \subset K$. Prove that $K = N \rtimes (K \cap H)$.

A4) Let F be a field and let G be the group of upper triangular matrices in $\mathrm{GL}(n, F)$.

(a) Prove that $G = U \rtimes D$, where U is the subgroup of G consisting of matrices with with 1's along the diagonal, and D is the group of diagonal matrices in $\mathrm{GL}(n, F)$.

(b) Suppose that $n = 2$. In this case note that $U \cong (F, +)$ and $D \cong (F^* \times F^*, \cdot)$. Describe the conjugation homomorphism $D \to \mathrm{Aut}(U)$ explicitly in terms of these isomorphisms (i.e., show how each element of $F^* \times F^*$ acts as an automorphism on F).

B1) We proved in class that A_5 is a simple group. Suppose that $n \geq 6$ and assume by induction that A_{n-1} is simple. Let N be a non-trivial proper normal subgroup of A_n.

(a) Show that $A_n = N \rtimes A_{n-1}$.

(b) Let N^* be the set of non-identity elements of N, and consider the action of A_{n-1} on N^* given by the conjugation homomorphism $A_{n-1} \to \mathrm{Aut}(N)$. Show that N^* is isomorphic as an A_{n-1}-set to $\{1, 2, \ldots, n-1\}$ with the standard action of A_{n-1}.

(c) An action of a group G on a set X is *triply transitive* if for every pair of triples (x_1, x_2, x_3) and (y_1, y_2, y_3) in $X \times X \times X$ with $x_i \neq x_j$ (respectively, $y_i \neq y_j$) when $i \neq j$, there exists some $g \in G$ such that $g \cdot x_i = y_i$ for $i = 1, 2, 3$. Show that A_{n-1} acts triply transitively on N^*.

(d) Derive a contradiction, and conclude that A_n is a simple group.

B2) Let N and H be groups. An *extension of N by H* is a group E along with a monomorphism $\phi : N \to E$ and an epimorphism $\psi : E \to H$ such that $\phi(N) = \mathrm{Ker}(\psi)$. In this case N embeds in E as a normal subgroup, with the quotient group being isomorphic to H. We identify N with its image under ϕ, and H with the quotient group E/N.

(a) We say that an extension E of N by H is a *split extension* if there is a homomorphism $\epsilon : H \to E$ (called the *splitting map* for the extension) such that $\psi \circ \epsilon$ is the identity map on H. Show that E is a split extension of N by H if and only if E is a semidirect product of N and H.

(b) Let $Q := \{\pm 1, \pm i, \pm j, \pm k\}$ denote the quaternion group of order 8. Show that Q can be realized as a non-trivial extension in four ways – thrice as an extension of C_4 by C_2, and once as an extension of C_2 by $C_2 \times C_2$, but that none of these extensions is split. This proves that Q cannot be written in a non-trivial way as a semidirect product.

Extra Credit Problems

C1) Prove that all groups of order less than 60 are solvable. [Hint: Show that if $|G| = n < 60$, then for some prime p dividing n, the number m of p-Sylow subgroups of G does not exceed 4. If $m > 1$, consider the action of G by conjugation on the set of all p-Sylow subgroups and obtain a non-trivial homomorphism $G \to S_m$.]

C2) Follow up on problem B2 of homework #6 and prove: if p is a prime with $p \geq 5$, then $\mathrm{PSL}(2, \mathbb{Z}/p\mathbb{Z})$ is a simple group. (This is the second infinite family of non-abelian finite simple groups; the alternating groups A_n for $n \geq 5$ were the first found. There is one overlap: $A_5 \cong \mathrm{PSL}(2, \mathbb{Z}/5\mathbb{Z})$).