Math 606 – Fall 2026 – Harry Tamvakis
PROBLEM SET 8 – Due November 5, 2026

1) On a Riemann surface X let $\mathfrak{D}$ denote the sheaf of divisors, that is, for any open set $U \subset X$, $\mathfrak{D}(U)$ consists of all maps $D : U \to \mathbb{Z}$ such that the support of D is locally finite. Prove that $\mathrm{H}^1(X, \mathfrak{D}) = 0$. [Hint: Imitate the proof that $\mathrm{H}^1(X, \mathcal{A}) = 0$ (where $\mathcal{A}$ is the sheaf of C^∞-functions) using a (discontinuous) integer-valued partition of unity.]

2) Let $\mathcal{O}^*$ be the sheaf of nowhere vanishing holomorphic functions and $\mathcal{M}^*$ denote the sheaf of meromorphic functions which are not identically zero on the Riemann surface X. Let $\alpha : \mathcal{O}^* \to \mathcal{M}^*$ be the natural inclusion map and $\beta : \mathcal{M}^* \to \mathfrak{D}$ be the map which assigns to each meromorphic function $f \in \mathcal{M}^*(U)$ its divisor $\mathrm{div}(f) \in \mathfrak{D}(U)$. Show that

$$0 \to \mathcal{O}^* \stackrel{\alpha}{\to} \mathcal{M}^* \stackrel{\beta}{\to} \mathfrak{D} \to 0$$

is an exact sequence of sheaves. Deduce that there is an exact sequence of abelian groups

$$0 \to \mathcal{O}^*(X) \to \mathcal{M}^*(X) \to Z^1(X) \to \mathrm{H}^1(X, \mathcal{O}^*) \to \mathrm{H}^1(X, \mathcal{M}^*) \to 0,$$

where $Z^1(X)$ is the group of all divisors on X.

3) Let D be a divisor of positive degree on a compact Riemann surface X and $L := L(D)$. Prove that (i) if $k > 0$ is sufficiently large, then L^k admits a non-trivial holomorphic section, while (ii) if $k < 0$ then L^k has no non-trivial holomorphic section.

4) In Problem Set 2 you showed that $\tan : \mathbb{C} \to \mathbb{P}^1 \smallsetminus \{-i, i\}$ is a covering map. Prove that the holomorphic 1-form $\dfrac{dz}{1+z^2}$, defined on $\mathbb{C} \smallsetminus \{-i, i\}$, can be extended to a holomorphic 1-form ω on $\mathbb{P}^1 \smallsetminus \{-i, i\}$. Compute the pullback form $\tan^* \omega$.

5) Let L be a holomorphic line bundle on a compact Riemann surface X and $m, n \geq 0$ be integers. The smooth (that is, C^∞) sections of the vector bundle $\Lambda^m(T^*_{\mathbb{C}}X)$ are differential m-forms on X; when $m = 0$, they are smooth functions $X \to \mathbb{C}$. For each point $a \in X$, the bilinear pairing

$$(\cdot\,, \cdot)_L : [\Lambda^m(T^*_{\mathbb{C}}X) \otimes L]_a \times [\Lambda^n(T^*_{\mathbb{C}}X) \otimes L^*]_a \to [\Lambda^{m+n}(T^*_{\mathbb{C}}X)]_a$$

is defined by

$$(\alpha, \beta)_L := \alpha_0 \wedge \beta_0 \cdot f(s),$$

where $\alpha = \alpha_0 \otimes s$ and $\beta = \beta_0 \otimes f$ with $\alpha_0, \beta_0 \in [T^*_{\mathbb{C}}X]_a$, $s \in L_a$, and $f \in L^*_a$. If α and β are smooth differential forms with values in L and L^*, respectively, then $(\alpha, \beta)_L$ is also smooth.

(a) Show that

$$\overline{\partial}((\alpha, \beta)_L) = (\overline{\partial}\alpha, \beta)_L + (-1)^{\deg \alpha}(\alpha, \overline{\partial}\beta)_L.$$

(b) Let $m \in \{0, 1, 2\}$, α be a smooth m-form with values in L, and β be a smooth $(2-m)$-form with values in L^*. Define

$$\langle \alpha, \beta \rangle := \int_X (\alpha, \beta)_L.$$

Suppose that $p \in \{0, 1\}$, α is of type $(p, 0)$, and β is of type $(1-p, 0)$. Show that

$$\langle \overline{\partial}\alpha, \beta \rangle + (-1)^p \langle \alpha, \overline{\partial}\beta \rangle = 0.$$

In particular, if α is a smooth section of L and β is a holomorphic 1-form with values in L^*, then $\langle \overline{\partial}\alpha, \beta \rangle = 0$.