Math 660 – Spring 2024 – Harry Tamvakis
PROBLEM SET 8 – Due April 11, 2024

1) Let Ω be an open set in $\mathbb{C}$, let $0 \in \Omega$, and let $f : \Omega \to \mathbb{C}$ be an injective holomorphic map with $f(0) = 0$. Let $R > 0$ be such that $\overline{D}(0, R) \subset \Omega$, and let C be the circle $t \mapsto Re^{2\pi it}$, $0 \leq t \leq 1$. Denote by $g : f(\Omega) \to \Omega$ the inverse of f.

(a) Prove that there exists a $\delta > 0$ such that

$$g'(w) = \frac{1}{2\pi i} \int_C \frac{dz}{f(z) - w}, \quad \text{for } |w| < \delta.$$

(b) Deduce that $g(w) = \displaystyle\sum_{n=1}^{\infty} b_n w^n$ with $nb_n = \displaystyle\frac{1}{2\pi i} \int_C \frac{dz}{(f(z))^n}$, $n \geq 1$.

(c) Show also that $b_n = \displaystyle\frac{1}{n!} \left(\frac{d}{dz}\right)^{n-1} (h(z))^n|_{z=0}$, where $h(z) = \displaystyle\frac{z}{f(z)}$.

2) Exercise 268 in the textbook.

3) Prove or disprove: There exists a sequence of polynomials P_n such that $P_n(0) = 1$ for all $n \geq 1$ but $\displaystyle\lim_{n\to\infty} P_n(z) = 0$ for every $z \neq 0$.

4) Let $a_1, \ldots, a_n$ be distinct points in $\mathbb{C}$, and let $\Omega = \mathbb{C} \smallsetminus \{a_1, \ldots, a_n\}$. Let $\mathfrak{U} = \{D_i\}_{i\in I}$ be an open covering of Ω by disks. Prove that $\mathrm{H}^1(\mathfrak{U}, \mathbb{C}) \cong \mathbb{C}^n$. Construct (with proof) an example of a domain Ω' in $\mathbb{C}$ and an open covering $\mathfrak{U}$ of Ω' such that $\mathrm{H}^1(\mathfrak{U}, \mathbb{C})$ is a complex vector space of infinite dimension.

5) Let X be a topological space and $\mathcal{F}$ be a sheaf of complex valued functions on X. Let $\mathfrak{U} := \{U_i\}_{i\in I}$ and $\mathfrak{V} := \{V_j\}_{j\in J}$ be two open coverings of X. Suppose that $\mathfrak{V}$ is a refinement of $\mathfrak{U}$ and let $\tau : J \to I$ be a map such that $V_j \subset U_{\tau(j)}$ for all $j \in J$. Consider the induced map

$$\tau^* : \mathrm{H}^1(\mathfrak{U}, \mathcal{F}) \to \mathrm{H}^1(\mathfrak{V}, \mathcal{F})$$

which sends $(f_{ij}) \in Z^1(\mathfrak{U}, \mathcal{F})$ to $(g_{\alpha\beta})$ where $g_{\alpha\beta} := f_{\tau(\alpha)\tau(\beta)}|_{V_\alpha \cap V_\beta}$ for each $\alpha, \beta \in J$. Prove that τ^* is *injective*.