Math 600 – Fall 2025 – Harry Tamvakis

PROBLEM SET 9 – Due November 13, 2025

A1) Let d be a non-zero square free integer, that is, d is divisible by no perfect square other than 1 (d might be negative). Consider one of the square roots of d in $\mathbb{C}$, and call it $\sqrt{d}$. Write $\mathbb{Z}[\sqrt{d}]$ for the set

$$\mathbb{Z}[\sqrt{d}] := \{x + y\sqrt{d} \mid x, y \in \mathbb{Z}\} \subset \mathbb{C}.$$

Check that under the ordinary operations in $\mathbb{C}$, $\mathbb{Z}[\sqrt{d}]$ is an integral domain (you don't need to write up the proof).

(a) Prove that $x + y\sqrt{d}$ is a *unit* in $\mathbb{Z}[\sqrt{d}]$ if and only if $x^2 - dy^2 = \pm 1$.

(b) If $d < 0$, show the number of units in $\mathbb{Z}[\sqrt{d}]$ is always finite and that the group of units is a finite cyclic group. Find this group explicitly if $d < 0$ for each d.

(c) Now consider the case $d \geq 2$ and start first with $d = 2$. Then $\xi = 1 + \sqrt{2}$ is a unit of $\mathbb{Z}[\sqrt{2}]$. Show that the group of units contains the cyclic group generated by ξ, which is an infinite cyclic group. Now choose $d \geq 2$, square free, and so that $d + 1$ *is* a square (e.g. $d = 15, 35, 143, \ldots$) Show that the group of units of $\mathbb{Z}[\sqrt{d}]$ is an infinite group (with more work one can show that the group of units is always infinite when $d \geq 2$).

A2) Let σ be an automorphism of the ring $\mathbb{R}$ of real numbers (i.e. σ is a ring isomorphism $\mathbb{R} \to \mathbb{R}$). Prove that $\sigma(x) = x$, for all $x \in \mathbb{R}$. In other words, the only automorphism of $\mathbb{R}$ is the identity. [Hint: First show that $\sigma(q) = q$ for any rational number q, and then that $x < y$ implies $\sigma(x) < \sigma(y)$ for all $x, y \in \mathbb{R}$.] By the way, this is *false* for $\mathbb{C}$; in fact, there are infinitely many automorphisms of $\mathbb{C}$.

A3) (a) An element x of a ring R is called *nilpotent* if $x^n = 0$ for some integer $n \geq 1$. Show that the set of nilpotent elements in a commutative ring R is an ideal of R.

(b) Find all nilpotent elements in the ring $(\mathbb{Z}/n\mathbb{Z}, +, \cdot)$.

A4) Let F be any field. Prove that the ring $M_n(F)$ of all $n \times n$ matrices with entries in F has no non-zero proper two sided ideal.

B1) (a) Let $R = C[0, 1]$ be the ring of continuous functions $f : [0, 1] \to \mathbb{R}$. Given $c \in [0, 1]$, let $M_c = \{f \in R \mid f(c) = 0\}$. Prove that M_c is a maximal ideal of R.

(b) Prove that if M is *any* maximal ideal of R then there is a real number $c \in [0, 1]$ such that $M = M_c$.

(c) Prove that M_c is not equal to the principal ideal generated by $x - c$. In fact, show that M_c is not a finitely generated ideal.

(d) Is a similar result to (b) true for the maximal ideals in the ring $C(\mathbb{R})$ of continuous functions $f : \mathbb{R} \to \mathbb{R}$? Justify your answer.

B2) Let F be a field and consider the ring $M_n(F)$ of $n \times n$ matrices over k. You showed in problem (A2) that $M_n(F)$ has no two-sided ideals other than (0) and $M_n(F)$. Let $B \subset M_n(F)$ be the subring consisting of all upper triangular matrices. In contrast to the full matrix ring $M_n(F)$, the ring B has plenty of two-sided ideals.

(a) Find all the maximal two-sided ideals of B. [Suggestion: Start with $n = 2$ and $n = 3$, where explicit calculations show what is happening. Then prove the general result.] You are expected to have a clean and clear description of these ideals and a proof that your description includes them all.

(b) Find *all* two-sided ideals of B.

B3) Let A be a commutative ring and $f(x) \in A[x]$, $f(x) \neq 0$. Suppose that there is a non-zero polynomial $g(x) \in A[x]$ such that $g(x)f(x) = 0$. Show that there is an $a \in A$ with $a \neq 0$ and $af(x) = 0$. Be careful, A may have nilpotent elements, i.e. elements α such that $\alpha^m = 0$.

<u>**Extra Credit Problem**</u>

C1) Let $F_2 = \langle a, b\rangle$ be the free group on two generators and

$$F_\infty = \langle x_1, x_2, x_3, \ldots\rangle$$

be the free group on countably many generators. Prove the surprising fact that there is a subgroup H of F_2 that is isomorphic to F_∞. That is, the free group on 2 generators contains the free group on infinitely many generators!