Math 606 – Fall 2026 – Harry Tamvakis
PROBLEM SET 9 – Due November 12, 2026

1) Let X be any Riemann surface. Prove that

$$0 \longrightarrow \mathbb{C}^* \longrightarrow \mathcal{O}^* \xrightarrow{\text{dlog}} \Omega \longrightarrow 0$$

is an exact sequence of sheaves on X, where $\text{dlog}(f) := \dfrac{df}{f}$.

2) Recall the operators $\partial : \mathcal{A}^0 \to \mathcal{A}^{1,0}$ and $\overline{\partial} : \mathcal{A}^0 \to \mathcal{A}^{0,1}$ defined locally by the equations

$$\partial f = \frac{\partial f}{\partial z}\, dz \qquad \text{and} \qquad \overline{\partial} f = \frac{\partial f}{\partial \overline{z}}\, d\overline{z}.$$

Let $\omega \in \mathcal{A}^2(X)$ be a smooth differential 2-form on the compact Riemann surface X. Prove that there exists a smooth function $f : X \to \mathbb{C}$ such that $\omega = \partial\overline{\partial} f$ if and only if $\displaystyle\int_X \omega = 0$.

3) Let X be a compact Riemann surface of genus g.

(a) Prove that if $g = 1$, then the canonical bundle K_X is trivial and $\dim \mathrm{H}^0(X, L) = \deg L$ for every holomorphic line bundle L of positive degree on X.

(b) Prove that if $g \geq 2$, then $\dim \mathrm{H}^0(X, K_X^{\otimes m}) = (2m-1)(g-1)$ for every integer $m \geq 2$.

4) Let ω be a meromorphic 1-form on the compact Riemann surface X with divisor E. Let D be any divisor on X and K be a canonical divsor. Prove that

$$2\left(h^0(K-D) - h^0(K-E+D)\right) = \deg(E-D) - \deg D.$$

5) Let $n \geq 1$ and $X \to \mathbb{P}^1$ be the Riemann surface of the algebraic function $w = \sqrt[n]{1-z^n}$, that is, the algebraic function defined by the polynomial $P(z,w) = w^n + z^n - 1$. Compute the genus of X.