Local behavior of Morse functions on Singular Spaces
By H. King

Goresky and MacPherson have developed a wonderful theory of Morse functions on certain stratified sets. The purpose of this talk is to indicate a slight generalization of their results and to drasticly shorten the proof of a key part of their work.

I will start with a brief review of their Morse theory on stratified sets. Since their theory only applies to Whitney stratified sets we must first review what a Whitney stratified set is. A Whitney stratified set is a smooth stratified set X contained in a smooth manifold M which satisfies certain properties. First of all, the strata of X must be smooth submanifolds of M. Next, if S and S' are two strata of X with S contained in the closure of S', then the tangent space of S' must have nice limits as one approaches S. For our purposes, we do not need to know what the precise condition is but we will give it anyway. Pick any point $y \in S$ and pick any coordinate chart around y. If x_i is a sequence of points in S' which converges to y and if y_i is a sequence of points in S which converges to y and if the line through x_i and y_i converges to a line τ and the tangent spaces $T_{x_i}S'$ converge to a subspace $Q \subset T_yM$ then we require that $\tau \subset Q$ and $T_yS \subset Q$. For example, let $M = \mathbb{R}^3$ and $X = \{(x,y,z) \in \mathbb{R}^3 | x^2 = zy^2\}$ and stratify X with two strata, $S = \{(0,0,z)\}$ and $S' = X - S$. Then X is not a Whitney stratified set because as we approach $(0,0,0)$ along the line $\{(0,y,0)\}$ the tangent space to S' is always the x, y plane. We can make X into a Whitney stratified set by refining the stratification- in particular by making $(0,0,0)$ be the 0-stratum. In fact, any algebraic set (or even any subanalytic set) has a Whitney stratification, so they are really quite general. The reason for dealing with Whitney stratified sets is that they behave nicely. The property we will use is that strata have tubular neighborhoods. We'll get more specific later on.

Now let us talk about Morse functions on Whitney stratified sets. Let X be a Whitney stratified set in a smooth manifold M. Let S be a stratum of Z, p a point in S and let $g: M \to \mathbb{R}$ be a smooth function. We say g is **normally nice** at p if $dg_p(Q) \neq 0$ for all limits Q of tangent spaces to strata of $X - S$. We say p is a **critical point** of $g|_X$ if $dg_p(T_pS) = 0$. Otherwise we say p is a **regular point** of $g|_X$. Those familiar with Goresky and MacPherson's Morse theory will recognize the normally nice condition as a slight generalization of nondepravity. A smooth map $g: M \to \mathbb{R}$ is a Morse function if $g|_X$ is proper and has distinct critical values, $g|_S$ is a (classical) Morse function for each stratum S of X, and g is normally nice at all $p \in X$.

Just as in the classical Morse theory on manifolds, Morse functions are generic and allow you to build up X from simple building blocks. In particular, suppose p is a critical point of $g|_X$, $c = g(p)$ and there are no other critical values of $g|_X$ in $[c-\epsilon, c+\epsilon]$. Let X_a denote $X \cap g^{-1}((-\infty, a])$. Then there is a pair of stratified sets (H_g, A_g) so that up to isomorphism, $X_{c+\epsilon}$ is obtained by attaching H_g to $X_{c-\epsilon}$ via some imbedding of A_g into $X_{c-\epsilon}$. This pair (H_g, A_g) (which is not uniquely defined) is called Morse data for $g|_X$ at p. If p is the only critical point in $g^{-1}(c)$ (for example if g is Morse), then this Morse data only depends on the local behavior of $g|_X$ near p.

In **[GM2]**, Goresky and MacPherson proved that for Morse functions (and in fact more generally for nondepraved functions) we have the product formula $(H_g, A_g) = (H_g^T, A_g^T) \times (H_g^N, A_g^N)$ where (H_g^T, A_g^T) is Morse data for $g|_S$ and (H_g^N, A_g^N) is Morse data for $g|_{D\cap X}$. Here S is the stratum containing p and D is a smooth submanifold of M which is transverse to S with $S \cap D = p$. Their proof is rather long. In this talk I will outline a much shorter proof. Full details will appear elsewhere, although they should not be too difficult to fill in. The basic idea is the following. Suppose that a neighborhood of p in X is isomorphic to a product $Z \times W$. Suppose furthermore that in these product coordinates, $g(z, w) = f(z) + h(w)$ for some functions $f: Z \to \mathbb{R}$ and $h: W \to \mathbb{R}$. Then under the mild assumption that the fibers of f and h be locally conelike, one can prove that Morse data for g is the product of Morse data for f and for h. Thus we only have to prove that a Morse function has such a product neighborhood.

This brings up an amusing idea for a definition of Morse function on any reasonable space such as a stratified set. Note that if M is a smooth manifold and g is a Morse function on M, then classical Morse theory tells us that at a critical point there is a product neighborhood and an f and an h as above so that f is the sum of squares of the coordinates and g is minus the sum of squares of the coordinates. We are then tempted to define a Morse critical point of a function $g: X \to \mathbb{R}$ to be a critical point which has a neighborhood of the form $Z \times W$ so that $g(z, w) = f(z) + h(w)$ and f has a local minimum and h has a local maximum. It is then not hard to show density of Morse functions on stratified sets (just keep approximating any function by a function which is Morse on neighborhoods of larger and larger skeleta, arbitrarily making the approximation a local minimum or maximum in direction normal to the strata). However, this definition has the defect that a function which is Morse in the sense of Goresky and MacPherson might not be Morse in this sense. For example, let $X = \{(x, y, z) \in \mathbb{R}^3 | z^2 = x^2 + y^2\}$ and let $g : \mathbb{R}^3 \to \mathbb{R}$ be $g(x, y, z) = z$. A little thought shows that this is Morse in the sense of Goresky and MacPherson, but not in the above sense.

We will now switch to a general discussion of isolated singularities of functions between spaces. Let $f: X \to Y$ be a morphism in some category CAT. We assume CAT is somehow topological, so X and Y are topological spaces with perhaps some kind of structure and f is a continuous map. For example, CAT could be smooth manifolds, topological manifolds, smooth stratified sets (of some sort) or topological stratified sets (more precisely, say Siebenmann's CS sets). A point $p \in X$ is called a **regular point** of f if there is a CAT imbedding $h : U \times V \to X$ onto a neighborhood of p where U is a neighborhood of $f(p)$ so that $fh(u, v) = u$. Notice for example that in the smooth manifold case the implicit function theorem implies that this is equivalent to the usual notion, namely that df_p be nonsingular. A point which is not regular is called a **critical point**. We say that f is a **submersion** if all points of X are regular. We will now restrict our category CAT to be nice. First of all we want the objects to be locally compact, locally conelike and metrizable. The other condition we want is more subtle. It is not worth being precise here, but the rule of thumb is that we want a proper CAT submersion to always be a bundle map. What we actually need is some local form of this, for example the Union Lemma 6.9 of **[S]** although I believe one can get by with something slightly weaker. Anyway, the four categories I mentioned above are nice. (I was not specific about the smooth stratified set category. What I have in mind is a certain category based on resolution of singularities which is about as general as I can think of without behaving too badly. It includes all Whitney stratified sets and Thom stratified sets.)

We will now study some results on isolated critical points which appeared in **[K1]**,**[K2]** and **[K3]**. These papers actually only do the manifold case explicitly, leaving the stratified set case to the reader, but the proofs for any nice category are word for word the same as the proofs presented there. A sketch of the proof appears at the end of this paper.

For simplicity we will restrict to the case where the target Y of f is $\mathbb{R}^k$. We will actually only use the case $k = 1$ here, although the $k = 2$ case proves some results contained in **[GM3]**. For $\epsilon > 0$ let $\epsilon B^k = \{y \in \mathbb{R}^k |\ |y| \leq \epsilon\}$ and let $\epsilon S^{k-1} = \{y \in \mathbb{R}^k |\ |y| = \epsilon\}$. Let p be an isolated critical point of $f: X \to \mathbb{R}^k$. We may as well suppose $f(p) = 0$. Then there is a $\delta > 0$ and maps $\varphi: U_1 \times [0,1] \to U$ and $\rho: (N - p) \times \delta B^k \to U$ satisfying the following properties:

a) U is a neighborhood of p in $X \cap f^{-1}(\delta B^k)$, $N = U \cap f^{-1}(0)$ and $U_1 = U \cap f^{-1}(\delta S^{k-1})$.

b) ρ is a CAT imbedding.

c) φ is an imbedding onto $U - N$ when restricted to $U_1 \times (0, 1]$.

d) $f\rho(x, y) = y$ for all $(x, y) \in (N - 0) \times \delta B^k$.

e) $f\varphi(x, t) = tf(x)$ for all $(x, t) \in U_1 \times [0, 1]$.

f) $\rho(x, 0) = x$ for all $x \in N - p$.

g) $\varphi(x, 1) = x$ for all $x \in U_1$.

h) $\rho(\varphi(x, 0), tf(x)) = \varphi(x, t)$ for all $t \in [0, 1]$ and $x \in U_1$ with $\varphi(x, 0) \neq p$.

i) $\varphi(\rho(x, y), t) = \rho(x, ty)$ for all $x \in N - 0$, $y \in \delta S^{k-1}$ and $t \in [0, 1]$.

The basic idea is that ρ is a trivialization of a tubular neighborhood of $N - p$ and φ shows that U is a mapping cylinder and furthermore, ρ and φ are compatible with f and each other as much as possible. (For those comparing this account with that in **[K2]**, I had combined φ and ρ into one function called p. I now find this confusing, and prefer using the two maps φ and ρ which contain all the information one needs.) Note that in case $k = 1$ one could take $(U, U \cap f^{-1}(-\delta))$ as Morse data.

As stated above, the properties of φ imply that U is a mapping cylinder of the map $\varphi|: U_1 \to N$ and that this mapping cylinder structure is intimately related to the behavior of f. So the local behavior of f is determined by $f|_{U_1}$ (which is just a fibration over δS^{k-1}) and the map $\varphi|: U_1 \to N$ (which fibers over $N - p$ and collapses the rest to 0). This a great help in understanding the behavior of f and can be used for example to classify isolated singularities. An even better description of the local behavior of f can be obtained if we know that N is conelike at p. In this case we can pick U to be a cone so that the cone structure is closely connected to f. In particular, for an object K in CAT, let $cK = K \times [0, 1]/K \times 0$ denote the cone on K. Of course, this might not be a CAT object (for example in the manifold case) but for simplicity we will pretend it is. Anyway, in the stratified set cases cK will be a CAT object. Now consider the case where N is isomorphic to cK for some CAT object K. Using the cone lines $x \times [0,1]$ in cK and the maps ρ and φ it is not hard to obtain an isomorphism $\theta: c(\partial U) \to U$ so that $f\theta(x,t) = tf(x)$ where $\partial U = U_1 \cup \rho((K \times 1) \times \delta B^k)$. (In the smooth case you must do a little corner rounding to ∂U at the corner $\rho((K \times 1) \times \delta S^{k-1})$.) Since this construction only appears in my unpublished thesis I will give it here. If $z \in U_1 - \rho((N-p) \times \delta S^{k-1})$ and $u \in (0,1]$, we let $\theta(z,u) = \varphi(z,u)$. If $z = \rho((x,s),y)$ for $x \in K$, $s \in (0,1]$ and $z \in \delta S^{k-1}$ we let $\theta(z,u) = \varphi(z,u)$ if $u^2 \geq s$ and we let $\theta(z,u) = \varphi(\rho((x,\sqrt{s}u),y),u)$ if $0 < u^2 \leq s$. If $z = \rho((x,1),y)$ we let $\theta(z,u) = \rho((x,u),uy)$. Again, in the smooth case you must smooth this a little.

The existence of θ has the following consequence- the behavior of f is determined by the behavior of $f|_{\partial U}$ which is relatively far away from the singularity. Furthermore $f|_{\partial U}$ is very nice. On U_1 it is a fiber bundle over δS^{k-1}. On the rest of ∂U it is a (trivial) bundle over δB^k with fiber K. Those familiar with Milnor's work on singular points of complex hypersurfaces see an old friend here – this is a generalization of the Milnor fibration. In effect, we have found 'spherical coordinates' in which the function f has a simple form. Using these spherical coordinates, it is now elementary to prove that in the case $k = 1$, the Morse data for the sum of two real valued functions is the product of their Morse data (assuming both inverse images of 0 are conelike).

The existence of θ also has consequences for families of isolated critical points. Suppose $f_z: X \to \mathbb{R}^k$, $z \in Z$ is a family of functions. Pick some $z_0 \in Z$. We will only be interested in the behavior of f_z for z near z_0 so in what follows we allow ourselves without notice to shrink Z to some neighborhood of z_0 to make our statements true. Suppose that p is an isolated critical point of each f_z and furthermore is 'uniformly isolated', i.e. for some neighborhood B of p, each f_z has no critical points in $B - p$. (In the terminology of **[K3]**, there is no coallescing). Then we can just parameterize our φ and ρ above to get continuous families φ_z and ρ_z. Since our ∂U_z's are far away from the critical point, they are all isomorphic also, in fact isomorphic via an isomorphism preserving the fibers of f_z. What we would really like is to parameterize θ. This is not always possible (see **[K3]**), but it is straightforward to see that a necessary and sufficient condition for the existence of a parameterized $\theta_z: c(\partial(U_{z_0})) \to X$ with $f_z\theta_z(x,t) = tf_{z_0}(x)$ is that there be a family of imbeddings $\lambda_z: cK \to X$ so that $\lambda_z(vertex) = p$ and so each image $\lambda_z(cK)$ is a neighborhood of p in $f_z^{-1}(0)$, i.e. the $f_z^{-1}(0)$'s are uniformly conelike at p. In particular, if the $f_z^{-1}(0)$'s are uniformly conelike at p then there is a family of imbeddings $h_z: U_{z_0} \to X$ so that $f_z h_z = f_{z_0}$, i.e. the family f_z is really trivial. This h_z is easy to construct: just let $h_z\theta_{z_0}(x,t) = \theta_z(x,t)$.

Now let us turn our attention to a normally nice critical point p of a smooth function $f: M \to \mathbb{R}$ restricted to a Whitney stratified set x. Let S be the stratum of X which contains p. Take a tubular neighborhood $\pi: T \to S$ of S. Define $g: T \to \mathbb{R}$ by $g = f - f\pi$. Now g also has a normally nice critical point at p since $d(f\pi)(Q) \subset df(\text{Image}(d\pi)) = df(T_pS) = 0$ for any subspace $Q \subset T_pM$, hence $df(Q) = dg(Q)$. By the theory of Whitney stratified sets there is an imbedding $\theta: V \times Y \to X$ so that $\pi\theta(z,y) = z$ where V is a neighborhood of p in S and $Y = X \cap \pi^{-1}(p)$ after perhaps shrinking the tubular neighborhood T. We thus get a family $g_z: Y \to \mathbb{R}$, $z \in V$ defined by $g_z(y) = g\theta(z,y)$. Normal nicety implies that this family g_z has uniformly isolated singularities. It also guarantees that $g^{-1}(0)$ is a Whitney stratified set. But then the theory of Whitney stratified sets implies that $g_z^{-1}(0)$ is uniformly locally conelike. Hence by the theory of singularities outlined above, there is a family of imbeddings $h_z: Y' \to Y$, $z \in V'$ so $g_z h_z = g_p$. Define $\theta': V' \times Y' \to X$ by $\theta'(z,y) = \theta(z, h_z(y))$. Then $g\theta'(z,y) = g\theta(z,h_z(y)) = g_z h_z(y) = g_p(y)$. But $f\pi\theta'(z,y) = f(z)$, so $f\theta'(z,y) = f(z) + g_p(y)$ and the function splits. Hence the Morse data for f is a product of the Morse data for $f|_S$ and g_p.

The reader familiar with **[GM2]** will notice the result here is slightly weaker than that in **[GM2]**. We only show that for some Morse data, the product formula holds. In **[GM2]** it is shown that for a nondepraved critical point there is a canonical choice of Morse data (corresponding to taking $U = \epsilon B^m \cap f^{-1}([-\delta,\delta])$) and the product formula holds for this canonical Morse data. It would be possible to refine the above proof to obtain this result also. In fact, by **[K2]** if one chooses the cone structure on N in which the cone levels are points equidistant from p then the canonical Morse data will just be the $(U, U \cap f^{-1}(-\delta))$ we obtain anyway. However, it is not really neccessary to do so. The reason is that by **[K2]**, any two conelike Morse data will be invertibly cobordant which implies they are isomorphic after crossing with $\mathbb{R}$. In particular they will have the same intersection homology groups (even for non-topologically invariant perversities if one so desired). Since the canonical Morse data is also conelike, any of the Morse data we construct will have intersection homology isomorphic to that of the canonical Morse data. Since **[GM1]** only cares about the intersection homology of the Morse data this is all we need.

We promised to sketch a proof of the existence of φ and ρ. After replacing X with some neighborhood of p we may assume that p is the only critical point of f. Let B_i, $i = 1, 2, \ldots$ be a sequence of compact neighborhoods of p so that $B_{i+1} \subset$ interior B_i and $p = \cap B_i$. In the manifold case, you could take the B_i's to be balls around p with radii approaching 0. Let $N_i = f^{-1}(0) \cap B_i -$ interior B_{i+2}. Since CAT is nice, N_1 has a neighborhood which fibers over some neighborhood δB^k of 0. Since any bundle over a ball is trivial, this means we have a CAT imbedding $\rho_1: N_1 \times \delta B^k \to X$ onto a neighborhood of interior N_i so that $f\rho_1(x,y) = y$ and $\rho_1(x,0) = x$. By taking a smaller δ if necessary, we may make sure that if S is the frontier of B_2 then $S \cap f^{-1}(\delta B^k) = S \cap \text{image } \rho_1$. Now let $U = (B_2 \cap f^{-1}(\delta B^k)) \cup \text{image } \rho_1$.

As above, we have an imbedding $\rho_2: N_2 \times \delta_2 B^k \to U$ with $\delta_2 < \delta$ which agrees with ρ_1 on the overlap $(N_1 \cap N_2) \times \delta_2 B^k$ so that $f\rho_2(x,y) = y$ and $\rho_2(x,0) = x$. In fact, letting $\delta_1 = \delta$, we can get a whole sequence of imbeddings $\rho_i: N_i \times \delta_i B^k \to U$ with $\delta_i < \delta_{i-1}$ so that $f\rho_i(x,y) = y$, $\rho_i(x,0) = x$ and ρ_i agrees with ρ_{i-1} on the overlap $(N_{i-1} \cap N_i) \times \delta_i B^k$. Let $A_i = \delta_i B^k -$ interior $\delta_{i+1} B^k$. Let $C_i = U \cap f^{-1}(A_i)$. Since CAT is nice, we know that $f: C_i \to A_i$ is a bundle map. But $A_i = S^{k-1} \times [0,1]$, hence each C_i is also a product with an interval. More precisely, letting $U_i = U \cap f^{-1}(\delta_i S^{k-1})$ we have an isomorphism $\varphi_i: U_i \times [\delta_{i+1}/\delta_i, 1] \to C_i$ so that $f\varphi_i(x,t) = tf(x)$ and $\varphi_i(x,1) = x$. Furthermore, by being careful we can also insure that $\varphi_i(\rho_j(x,y),t) = \rho_j(x,ty)$ for all $j \leq i$, $x \in N_j - N_{j+1}$, $y \in \delta_j S^{k-1}$ and $t \in [\delta_{i+1}/\delta_i, 1]$. Note we have isomorphisms $\kappa_i: U_1 \to U_i$ defined inductively by $\kappa_i = \varphi_{i-1}(\kappa_{i-1}(x), \delta_i/\delta_{i-1})$. Let $N = U \cap f^{-1}(0)$. We can now define an isomorphism $\varphi: U_1 \times (0,1] \to U - N$ by letting $\varphi(x,t) = \varphi_i(\kappa_i(x), t/\delta_i)$ if $\delta_{i-1} \leq t \leq \delta_i$. We can also define an imbedding $\rho: (N-0) \times \delta B^k \to U$ by setting $\rho(x,y) = \rho_i(x,y)$ if $x \in N_i - N_{i+1}$ and $|y| \leq \delta_i$ and by setting $\rho(x,y) = \varphi(\kappa_i^{-1}(\rho(x, \delta_i y/|y|)), |y|/\delta)$ if $x \in N_i - N_{i+1}$ and $|y| \geq \delta_i$. Note that $f\rho(x,y) = y$, $f\varphi(x,t) = tf(x)$, $\rho(x,0) = x$, $\varphi(x,1) = x$ and $\varphi(\rho(x,y),t) = \rho(x,ty)$ if $y \in \delta S^{k-1}$. It remains to show that φ extends continuously to a map $\varphi: U_1 \times [0,1] \to U$. Suppose $x = \rho(z,y) \in U_1$, then $\lim_{t\to 0} \varphi(x,t) = z$. Now suppose $x \in U_1 - \text{image } \rho$. Let z be a limit point of $\varphi(x,t)$ as $t \to 0$. Since $f\varphi(x,t) = tf(x)$ we know that $f(z) = 0$. But if $z \neq p$ then we must have $x = \rho(z, f(x))$, a contradiction. So $\lim_{t\to 0} \varphi(x,t) = p$. Thus φ extends continuously.

[GM1] M. Goresky and R. MacPherson, Morse Theory I.

[GM2] M. Goresky and R. MacPherson, Morse Theory II.

[GM3] M. Goresky and R. MacPherson, Morse Theory III.

[K1] H. King, Thesis, Univ. of Ca., Berkeley (1974), unpublished.

[K2] H. King, Topological Type of Isolated Critical Points, Annals of Math., Vol. 107 (1978), 385-397.

[K3] H. King, Topological Type in Families of Germs, Inventiones Math., Vol. 62 (1980), pp. 1-13.

[S] L.Siebenmann, Deformation of Homeomorphisms on Stratified Sets, Comm. Math. Helv., Vol. 47 (1972), pp. 123-163.