TOPOLOGY OF ISOLATED CRITICAL POINTS OF FUNCTIONS ON SINGULAR SPACES

Henry C. King

Universty of Maryland

In [K1], [K2] and [K3] isolated critical points of functions between manifolds were studied and it was shown that the topological right and right-left equivalence classes were determined by an analogue of the Milnor fibration. It was asserted that similar results hold for functions between stratified sets, and in fact the proofs are almost word for word the same. In this paper, we will look at these results in the context of singular spaces. We also give an application of these results to Goresky's and MacPherson's theory of Morse functions on Whitney stratified sets.

Although quite powerful, these results hold in a fair amount of generality so we do not wish to restrict our spaces too much. Let $f: X \to Y$ be a morphism in some category CAT. We assume CAT is somehow topological, so X and Y are topological spaces with perhaps some kind of structure and f is a continuous map. For example, CAT could be:

1. smooth manifolds and smooth maps,
2. topological manifolds and continuous maps,
3. smooth stratified sets (of some sort) and maps (of some sort).†
4. topological stratified sets (more precisely, say Siebenmann's CS sets) and continuous maps which map strata to strata.

Definitions: A point $p \in X$ is called a *regular point* of f if there is a neighborhood U of $f(p)$ in Y, a CAT object V and a CAT imbedding $h: U \times V \to X$ onto a neighborhood of p so that $fh(u, v) = u$. In other words a regular point is where the function is locally a projection. Notice for example that in the smooth manifold case the implicit function theorem implies that this is equivalent to the usual notion, namely that df_p be nonsingular. A point which is not regular is called a *critical point* of f. A point $x \in X$ is an *isolated critical point* of f if there is a neighborhood U of x in X so that all points of $U - x$ are regular points of f. We say that f is a *submersion* if all points of X are regular points of f.

We will now restrict our category CAT to be nice. The reader who wishes can skip this paragraph, content in the knowledge that the four categories above are nice. We have already implicitly made one assumption about our category. In defining a regular point we needed a CAT structure on $U \times V$. Thus we must assume that if U and V are CAT objects then there is a natural way to put a CAT

† I am intentionally not specific about the smooth stratified set category. What I have in mind is a certain category based on resolution of singularities which is about as general as I can think of without behaving too badly. It includes all Thom stratified sets.

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

structure on the Cartesian product $U \times V$ of the underlying topological spaces. Furthermore we want the projections to U and V to be CAT maps. We need an open subset of a CAT object to inherit a natural CAT structure. Next we need to have the reals $\mathbb{R}$ in CAT, so that we can define the cone on CAT object. We want CAT objects to be locally compact, locally conelike and metrizable. The other condition we want is more subtle. It is not worth being precise here, but the rule of thumb is that we want a proper CAT submersion to always be a bundle map. What we actually need is some local form of this, for example the Union Lemma 6.9 of [S] although I believe one can get by with something slightly weaker. For some results we also need an isotopy extension theorem, e.g., 6.6 of [S].

The point of this last subtle condition is that it provides you with the main technical tool used in [K2] and [K3]. This allows one to get results for very general types of spaces. It is essentially a replacement for the technique of integrating vector fields.

In fact, those readers only interested in spaces such as manifolds or Whitney stratified sets where vector field techniques are available can ignore all this Union Lemma 6.9 mumbo jumbo. The results in [K2] and [K3] can all be proven fairly readily in this context using familiar vector field techniques. For example, this was the only technique used in [K1].

We will now study some results on isolated critical points which appeared in [K1],[K2] and [K3]. These papers actually only do the manifold case explicitly, leaving the stratified set case to the reader, but the proofs for any nice category are word for word the same as the proofs presented there. A sketch of the proof appears at the end of this paper.

So let $f\colon X \to Y$ be a CAT morphism. Let p be an isolated critical point of f. Since Y is locally conelike and we are only interested in local behavior, we may as well assume that Y is the cone $\mathfrak{c}S$ on some compact CAT object S and that $f(p)$ is the vertex $\mathfrak{v}$ of the cone.

Let us digress a moment to establish some notation about cones. If Z is a topological space then $\mathfrak{c}Z$ denotes the cone on Z which we define to be the quotient space $Z \times [0,1]/Z \times 0$. The point to which $Z \times 0$ collapses is called the *vertex* of the cone and is denoted by $\mathfrak{v}$. If $x \in \mathfrak{c}Z$ and $t \in [0,1]$ then tx is defined as follows. We know that $x = (z,s)$ for some $(z,s) \in Z \times [0,1]$. Then we let $tx = (z, st)$. If $\epsilon \in (0,1]$ then $\epsilon\mathfrak{c}Z$ denotes the subset $Z \times [0,\epsilon]/Z \times 0 \subset \mathfrak{c}Z$. We sometimes use the open cone $\breve{\mathfrak{c}}Z = Z \times [0,1)/Z \times 0$. The open cone on Z minus the vertex is homeomorphic to $Z \times \mathbb{R}$ and thus has a natural CAT structure. Sometimes (for example in the stratified set categories) this extends to a CAT structure on the whole open cone. For simplicity in stating results, we will pretend one can do such an extension, but this is not really necessary. We will also pretend the CAT structure extends from the open cone to the closed cone (by allowing CAT objects to have boundary), but again this is not really necessary. Eliminating these simplifications would only add a layer of technicality which would obscure the main ideas.

I expect that in most applications, $Y = \mathbb{R}^k$. In this case we could take S to be a sphere of dimension $k-1$. Then $\epsilon\mathfrak{c}S$ would be the ball of radius ϵ. Actually I imagine most people would only be interested in the cases $k=1$ or 2 so in fact the reader can think of S as being either two points or a circle. For the application to Morse theory, we only need $k=1$ and then S is just two points.

By the argument in [K2], there is a $\delta > 0$ and maps $\varphi: U_1 \times [0,1] \to U$ and $\rho: (N-p) \times \delta\mathfrak{c}S \to U$ satisfying the following properties:

a) U is a neighborhood of p in $X \cap f^{-1}(\delta\mathfrak{c}S)$, $N = U \cap f^{-1}(\mathfrak{v})$ and $U_1 = U \cap f^{-1}(S \times \delta)$.

b) ρ is a CAT imbedding.

c) φ is an imbedding onto $U - N$ when restricted to $U_1 \times (0,1]$.

d) $f\rho(x,y) = y$ for all $(x,y) \in (N-p) \times \delta\mathfrak{c}S$.

e) $f\varphi(x,t) = tf(x)$ for all $(x,t) \in U_1 \times [0,1]$.

f) $\rho(x,\mathfrak{v}) = x$ for all $x \in N - p$.

g) $\varphi(x,1) = x$ for all $x \in U_1$.

h) $\rho(\varphi(x,0), tf(x)) = \varphi(x,t)$ for all $t \in [0,1]$ and $x \in U_1$ with $\varphi(x,0) \neq p$.

i) $\varphi(\rho(x,y),t) = \rho(x,ty)$ for all $x \in N-p$, $y \in S \times \delta$ and $t \in [0,1]$.

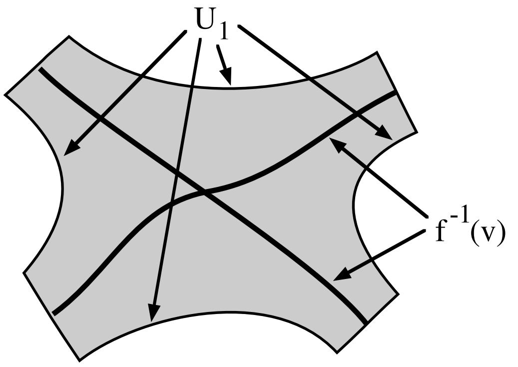

Figure 1

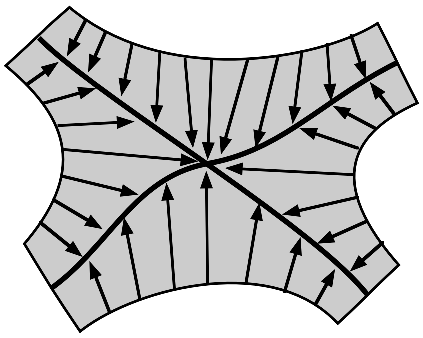

Figure 2

Figure 1 shows a picture of U with U_1 and $f^{-1}(\mathfrak{v})$ indicated. Figure 2 shows what the map φ does.

The basic idea is that ρ is a trivialization of a tubular neighborhood of $N-p$ and φ shows that U is a mapping cylinder and furthermore, ρ and φ are compatible with f and each other as much as possible. (For those comparing this account with that in [K2], I had combined φ and ρ into one function called p. I now find this confusing, and prefer using the two maps φ and ρ which contain all the information one needs.)

I will sketch a proof of the existence of ρ and φ at the end of this paper, but meanwhile we can look at a proof of the special case where $Y = \mathbb{R}$ and X is a Whitney stratified set. Assume that $f(p) = 0$. Pick a neighborhood V of p in X so that f submerses $V - p$. Then we may pick a controlled vector field v on $V - p$ so that $f_*(v) = \partial/\partial t$ where $\partial/\partial t$ is the standard unit vector field on $\mathbb{R}$. Integrate v to obtain a flow Φ. Then $\rho(x,y) = \Phi(x,y)$ and $\varphi(x,t) = \Phi(x,(t-1)f(x)) = \Phi(x, \pm(t-1)\delta)$. (Strictly speaking, $\varphi(x,0) = \lim_{t\to 0} \Phi(x,(t-1)f(x))$ since $\Phi(x,-f(x))$ might not be defined for some x. Of course, for those x, we have $\varphi(x,0) = p$.) Then by picking U and δ small enough, the properties a)-i) above just derive from the standard properties of flows.

As stated above, the properties of φ imply that U is a mapping cylinder of the map $\varphi|: U_1 \to N$ and that this mapping cylinder structure is intimately related to the behavior of f. So the local behavior of f is determined by $f|_{U_1}$ (which is just a fibration over $S \times \delta$) and the map $\varphi|: U_1 \times 0 \to N$ (which fibers over $N - p$ and collapses the rest to p). This a great help in understanding the behavior of f and can be used for example to classify isolated singularities.

An even better description of the local behavior of f can be obtained if we know that N is conelike at p. So suppose N is isomorphic to $\mathfrak{c}K$ for some CAT object K. Using the cone lines $x \times [0,1]$ in $\mathfrak{c}K$ and the maps ρ and φ it is not hard to obtain an isomorphism $\theta: \mathfrak{c}(\partial U) \to U$ so that $f\theta(x,t) = tf(x)$ where $\partial U = U_1 \cup \rho((K \times 1) \times \delta\mathfrak{c}S)$. (In the smooth case you must do a little corner rounding to ∂U at the corner $\rho((K \times 1) \times S \times \delta)$.) Since this construction only appears in my unpublished thesis I will give it here. If $z \in U_1 - \rho((N-p) \times S \times \delta)$ and $u \in (0,1]$, we let $\theta(z,u) = \varphi(z,u)$. If $z = \rho((x,s),y)$ for $x \in K$, $s \in (0,1]$ and $y \in S \times \delta$ we let $\theta(z,u) = \varphi(z,u)$ if $u^2 \geq s$ and we let $\theta(z,u) = \varphi(\rho((x,\sqrt{s}u),y),u)$ if $0 < u^2 \leq s$. If $z = \rho((x,1),y)$ we let $\theta(z,u) = \rho((x,u),uy)$. Again, in the smooth case you must smooth this a little. Figure 3 shows some of the cone lines of θ.

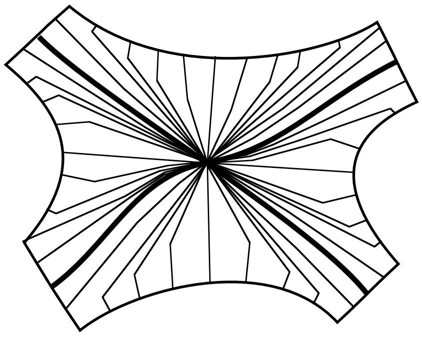

Figure 3

The existence of θ has the following consequence: the behavior of f is determined by the behavior of $f|_{\partial U}$ which is relatively far away from the singularity. Furthermore $f|_{\partial U}$ is very nice. On U_1 it is a fiber bundle over $S \times \delta$. On the rest of ∂U it is a (trivial) bundle over $\delta\mathfrak{c}S$ with fiber K. Those familiar with Milnor's work on singular points of complex hypersurfaces see an old friend here – this is a generalization of the Milnor fibration. In effect, we have found 'spherical coordinates' in which the function f has a simple form.

We can summarize these results as:

Theorem 1. *If $f^{-1}f(p)$ is locally conelike at an isolated critical point p, then there is an analogue of the Milnor fibration which determines the topological right equivalence class of f, i.e., determines f up to continuous change of coordinates in X. In particular, there is a set $T \subset X$ and an imbedding $\theta: (\mathfrak{c}T, \mathfrak{v}) \to (X,p)$ so that $f(x,t) = tf(x)$ for all $(x,t) \in \mathfrak{c}T$. Furthermore, $f|_T$ has a particularly simple form.*

One can be more precise as in [K2] and determine exactly when two Milnor fibrations give right or right-left equivalent germs. For right equivalence it is exactly the same invertible cobordism equivalence relation as in [K2]. For right-left equivalence, it is a little more complicated than in [K2] since S can be more complicated than a sphere. There is a group $\mathcal{B}(S)$ of homotopy equivalences of S which come from invertible cobordisms of S to itself. The group acts on Milnor fibrations via the induced fibration. Then two germs are topologically right-left equivalent if and only if their Milnor fibrations are in the same orbit of this action. For S a sphere the group $\mathcal{B}(S)$ is $\mathbb{Z}/2\mathbb{Z}$ generated by reflection, so this coincides with the result in [K2].

Specializing Theorem 1 to the case $Y = \mathbb{R}$ we see that the function f is determined up to right equivalence by the pair of spaces $(\partial U, \partial U \cap f^{-1}(f(p) - \delta))$. We call this pair the *Milnor data* for f at p. Let us see why the Milnor data determines the function. If (A, B) is the Milnor data for f at p, then ∂B has a collar $\partial B \times [-\delta, \delta]$ in $A - \operatorname{Int} B$. Define $g: A \to \mathbb{R}$ by $g(x) = -\delta$ for $x \in B$, $g(x) = t$ if $x = (y, t)$ in the collar, and $g(x) = \delta$ otherwise. Then by Theorem 1, there is an imbedding $h: \mathfrak{c}A \to X$ onto a neighborhood of p so that $fh(x,t) = tg(x) + f(p)$.

The existence of θ also has consequences for families of isolated critical points, as was seen in [K3]. Suppose $f_z: X \to \mathfrak{c}S$, $z \in Z$ is a family of functions. Pick some $z_0 \in Z$. We will only be interested in the behavior of f_z for z near z_0 so in what follows we allow ourselves without notice to shrink Z to some neighborhood of z_0 to make our statements true. Suppose that $f_z(p) = \mathfrak{v}$ for all $z \in Z$, p is an isolated critical point of each f_z and furthermore is 'uniformly isolated', i.e., for some neighborhood B of p, the map $F: (B - p) \times Z \to Y \times Z$ given by $F(x, z) = (f_z(x), z)$ has no critical points. (In the terminology of [K3], there is no coalescing†). Then we can just parameterize our φ and ρ above to get continuous families φ_z and ρ_z. Since our ∂U_z's are far away from the critical point, they are all isomorphic also, in fact isomorphic via an isomorphism preserving the fibers of f_z. What we would really like is to parameterize θ. This is not always possible (see [K3]), but it is straightforward to see that a necessary and sufficient condition for the existence of a parameterized $\theta_z: \mathfrak{c}(\partial(U_{z_0})) \to X$ with $f_z\theta_z(x,t) = tf_{z_0}(x)$ is that there be a family of imbeddings $\lambda_z: \mathfrak{c}K \to X$ so that $\lambda_z(\mathfrak{v}) = p$ and so each image $\lambda_z(\mathfrak{c}K)$ is a neighborhood of p in $f_z^{-1}(0)$, i.e., the $f_z^{-1}(\mathfrak{v})$'s are uniformly conelike at p. In particular, if the $f_z^{-1}(\mathfrak{v})$'s are uniformly conelike at p then there is a family of imbeddings $h_z: U_{z_0} \to X$ so that $f_z h_z = f_{z_0}$, i.e., the family f_z is really trivial. This h_z is easy to construct: just let $h_z\theta_{z_0}(x,t) = \theta_z(x,t)$.

Summarizing, we have:

Theorem 2. *Let $f_z: (X, p) \to (\mathfrak{c}S, \mathfrak{v})$ for $z \in Z$ be a family of functions with uniformly isolated critical points. Then f_z is topologically trivial if and only if $f_z^{-1}(\mathfrak{v})$ is topologically trivial.*

Suppose we have two functions $f: X \to \mathbb{R}$ and $g: Z \to \mathbb{R}$. Then we have their sum $f + g: X \times Z \to \mathbb{R}$ given by $(f+g)(x,z) = f(x) + g(z)$. Suppose that f and g have isolated critical points at p and q. Then it is easy to see that $f + g$ has an isolated critical point at (p, q). Suppose furthermore that $f^{-1}f(p)$ and $g^{-1}g(q)$ are both locally conelike. Then it is an elementary exercise to describe the Milnor data for $f + g$ in terms of that for f and for g. We have:

† Note that the definition of no coalescing given there is incorrect and the above definition should be used. Of course the two definitions are the same in the smooth category, but are probably not the same in the topological category.

Theorem 3. *If (A, A_-) is Milnor data for f at p and (B, B_-) is Milnor data for g at q then $(A \times \mathfrak{c}B \cup \mathfrak{c}A \times B, A_- \times \mathfrak{c}B \cup \mathfrak{c}A \times B_-)$ is Milnor data for $f + g$ at (p, q).*

For example, if $X = \mathbb{R}^n$, $Z = \mathbb{R}^k$, $f(x) = |x|^2$ and $g(z) = -|z|^2$, then Milnor data for f at 0 is $(S^{n-1}, \emptyset)$ and Milnor data for g at 0 is (S^{k-1}, S^{k-1}). Hence Milnor data for $f + g$ is $(S^{n-1} \times B^k \cup B^n \times S^{k-1}, B^n \times S^{k-1})$ which is exactly what we know from classical Morse theory.

We shall define below Morse data for an isolated critical point. If (A, A_-) is Milnor data for f then $(\mathfrak{c}A, A_- \times 1)$ is Morse data for f.

We may now restate Theorem 3 in terms of Morse data rather than Milnor data. It becomes a bit simpler, and also applies to the case where $f^{-1}f(p)$ and $g^{-1}g(p)$ are perhaps not locally conelike.

Theorem 4. *If (C, C_-) is Morse data for f at p and (D, D_-) is Morse data for g at q then $(C, C_-) \times (D, D_-)$ is Morse data for $f + g$ at (p, q).*

For example, if $Z = \mathbb{R}$ and $g(t) = t$, then Morse data for g at 0 is $([1, -1], \{-1\})$. Hence Morse data for $f + g$ is $(C \times [-1, 1], C \times \{-1\} \cup C_- \times [-1, 1])$. For another example, if $Z = \mathbb{R}$ and $g(t) = t^2$ then Morse data for g at 0 is $([-1, 1], \emptyset)$. Then Morse data for $f + g$ is $(C \times [-1, 1], C_- \times [-1, 1])$.

Now I will switch attention to Goresky's and MacPherson's Morse theory on Whitney stratified sets. The paper [K4] has more details on some of the results I state here.

Definitions: Let X be a Whitney stratified set in a smooth manifold M. Let S be a stratum of X, p a point in S and let $g\colon M \to \mathbb{R}$ be a smooth function. We say g is *normally nice* at p if $dg_p(Q) \neq 0$ for all limits Q of tangent spaces to strata of $X - S$. We say p is a *critical point* of $g|_X$ if $dg_p(T_pS) = 0$. Otherwise we say p is a *regular point* of $g|_X$.

Those familiar with Goresky and MacPherson's Morse theory will recognize the normally nice condition. Also as an exercise, the reader can prove using Thom's first isotopy lemma that the notions of critical and regular point coincide with the general definitions given previously for any reasonable choice of category.

Definition: A smooth map $g\colon M \to \mathbb{R}$ is a *Morse function* if $g|_X$ is proper and has distinct critical values, $g|_S$ is a (classical) Morse function for each stratum S of X, and g is normally nice at all $p \in X$.

Just as in the classical Morse theory on manifolds, Morse functions allow you to build up X from simple building blocks. In particular, suppose p is a critical point of $g|_X$, $c = g(p)$ and there are no other critical values of $g|_X$ in $[c - \epsilon, c + \epsilon]$. Let X_a denote $X \cap g^{-1}((-\infty, a])$. Then there is a pair of stratified sets (H_g, A_g) so that up to isomorphism, $X_{c+\epsilon}$ is obtained by attaching H_g to $X_{c-\epsilon}$ via some imbedding of A_g into $X_{c-\epsilon}$. This pair (H_g, A_g) (which is not uniquely defined) is called *Morse data* for $g|_X$ at p. If p is the only critical point in $g^{-1}(c)$ (for example if g is Morse), then this Morse data only depends on the local behavior of $g|_X$ near p.

In fact this Morse data derives from the Milnor data we defined above. If (A, A_-) is the Milnor data for $g|_X$ at p, then $(\mathfrak{c}A, A_-)$ is Morse data for $g|_X$ at p. To see this, recall that we had a neighborhood U of p in X and a $\delta > 0$. We may assume $\delta \leq \epsilon$. Then $(A, A_-) = (\partial U, \partial U \cap g^{-1}(c - \delta))$ and $U = \mathfrak{c}(\partial U)$. So we are asserting we may set $H_g = U$ and $A_g = \partial U \cap g^{-1}(c-\delta)$. The idea is that $X \cap g^{-1}([c-\delta, c+\delta]) - U$ is isomorphic to $((X-U) \cap g^{-1}(c-\delta)) \times [0,1]$ and hence does not contribute any part of the change from $X_{c-\delta}$ to $X_{c+\delta}$.

In [GM], Goresky and MacPherson proved that for Morse functions (and in fact more generally for nondepraved functions) we have the product formula $(H_g, A_g) = (H_g^T, A_g^T) \times (H_g^N, A_g^N)$ where (H_g^T, A_g^T) is Morse data for $g|_S$ and (H_g^N, A_g^N) is Morse data for $g|_{D\cap X}$. Here S is the stratum containing p and D is a smooth submanifold of M which is transverse to S with $S \cap D = p$. Their proof is rather long, it occupies much of the book.

Note however that there is a strong resemblance of their result to Theorem 4. We thus get a quick proof of their result if we can split g into the sum of two functions and then apply Theorem 4. In particular, let W be a small neighborhood of p in S and let Z be a small neighborhood of p in $D \cap X$. From the theory of Whitney stratified sets, we know that a neighborhood of p in X is isomorphic to a product $W \times Z$. Suppose that in these product coordinates, $g(w,z) = f(w) + h(z)$ for some functions $f: W \to \mathbb{R}$ and $h: Z \to \mathbb{R}$. Then under the mild assumption that the fibers of f and h be locally conelike, Theorem 4 shows that Morse data for g is the product of Morse data for f and for h. Thus we only have to prove that p has such a product neighborhood.

This brings up an amusing idea for a definition of Morse function in any nice category. Note that if M is a smooth manifold and g is a Morse function on M, then classical Morse theory tells us that at a critical point there is a product neighborhood and an f and an h as above so that f is the sum of squares of the coordinates and h is minus the sum of squares of the coordinates. We are then tempted to define a Morse critical point of a function $g: X \to \mathbb{R}$ to be a critical point which has a neighborhood of the form $W \times Z$ so that $g(w,z) = f(w) + h(z)$ and f has a local minimum and h has a local maximum. It is then not hard to show C^0 density of Morse functions on stratified sets (just keep approximating any function by a function which is Morse on neighborhoods of larger and larger skeleta, arbitrarily making the approximation a local minimum or maximum in direction normal to the strata). However, this definition has the defect that a function which is Morse in the sense of Goresky and MacPherson might not be Morse in this sense. For example, let $X = \{(x,y,z) \in \mathbb{R}^3 | z^2 = x^2 + y^2\}$ and let $g: \mathbb{R}^3 \to \mathbb{R}$ be $g(x,y,z) = z$. A little thought shows that this is Morse in the sense of Goresky and MacPherson, but not in the above sense.

We now prove a somewhat stronger version of the result of [GM].

Theorem 5. *Let $f: (M,p) \to (\mathbb{R}, 0)$ be a smooth function, let $X \subset M$ be a locally closed Whitney stratified set and suppose p is a normally nice isolated critical point of $f|_X$. Let S be the stratum of X which contains p. Then Morse data for f is homeomorphic to the product of Morse data for $f|_S$ and for $f|_{X\cap D}$ where D is any submanifold of M transverse to S with $D \cap S = p$.*

Proof: Take a tubular neighborhood $\pi: T \to S$ of S with $\pi^{-1}(p) \subset D$. Define $g: T \to \mathbb{R}$ by $g = f - f\circ\pi$. Now g also has a normally nice critical point at p since $d(f\circ\pi)(Q) \subset df(\text{image }(d\pi)) = df(T_pS) = 0$ for any subspace $Q \subset T_pM$, hence $df(Q) = dg(Q)$. By the theory of Whitney stratified sets there is an imbedding $\theta: V \times Y \to X$ so that $\pi\theta(z,y) = z$ where V is a neighborhood of p in S and

$Y = X \cap \pi^{-1}(p)$ after perhaps shrinking the tubular neighborhood T. We thus get a family $g_z: Y \to \mathbb{R}$, $z \in V$ defined by $g_z(y) = g\theta(z, y)$. Normal nicety implies that this family g_z has uniformly isolated singularities. It also guarantees that $g^{-1}(0)$ is a Whitney stratified set. But then the theory of Whitney stratified sets implies that $g_z^{-1}(0)$ is uniformly locally conelike. Hence by Theorem 2, there is a family of imbeddings $h_z: Y' \to Y$, $z \in V'$ so $g_z h_z = g_p$. Define $\theta': V' \times Y' \to X$ by $\theta'(z, y) = \theta(z, h_z(y))$. Then $g\theta'(z, y) = g\theta(z, h_z(y)) = g_z h_z(y) = g_p(y)$. But $f\pi\theta'(z, y) = f(z)$, so $f\theta'(z, y) = f(z) + g_p(y)$ and the function splits. Hence the Morse data for f is a product of the Morse data for $f|_S$ and g_p. $\square$

The reader familiar with [GM] will notice the result here is slightly weaker than that in [GM]. We only show that for some Morse data, the product formula holds. In [GM] it is shown that for a nondepraved critical point there is a canonical choice of Morse data (corresponding to taking $U = \epsilon B^m \cap f^{-1}([-\delta, \delta])$) and the product formula holds for this canonical Morse data. It would be possible to refine the above proof to obtain this result also, the reader interested in details can refer to [K4]. In fact, by [K2] if one chooses the cone structure on N in which the cone levels are points equidistant from p then the canonical Morse data will just be the $(U, U \cap f^{-1}(-\delta))$ we obtain anyway. However, it is not really necessary to do so. The reason is that by [K2], any two conelike Morse data will be invertibly cobordant which implies they are isomorphic after crossing with $\mathbb{R}$. In particular they will have the same intersection homology groups (even for non-topologically invariant perversities if one so desired). Since the canonical Morse data is also conelike, any of the Morse data we construct will have intersection homology isomorphic to that of the canonical Morse data. Since [GM] only cares about the intersection homology of the Morse data this is all we need.

David Trotman has pointed out that much of the proof of Theorem 5 carries over to the general situation where X is only assumed to be a Bekka stratified set, not necessarily Whitney stratified. In fact the only place where the proof might possibly fail is in the assertion that $g^{-1}(0)$ is Bekka stratified. If one could somehow show that $g^{-1}(0)$ were topologically trivial along S then Theorem 5 would be true in the case where X is Bekka stratified.

We promised to sketch a proof of the existence of φ and ρ. After replacing X with some neighborhood of p we may assume that p is the only critical point of f. Let B_i, $i = 1, 2, \ldots$ be a sequence of compact neighborhoods of p so that $B_{i+1} \subset \mathrm{Int} B_i$ and $p = \cap B_i$. In the manifold case, you could take the B_i's to be balls around p with radii approaching 0. Let $N_i = f^{-1}(\mathfrak{v}) \cap B_i - \mathrm{Int} B_{i+2}$. Since CAT is nice, N_1 has a neighborhood which fibers over some neighborhood $\delta\mathfrak{c}S$ of $\mathfrak{v}$. Since any bundle over a cone is trivial, this means we have a CAT imbedding $\rho_1: N_1 \times \delta\mathfrak{c}S \to X$ onto a neighborhood of $\mathrm{Int} N_i$ so that $f \circ \rho_1(x, y) = y$ and $\rho_1(x, \mathfrak{v}) = x$. By taking a smaller δ if necessary, we may make sure that if S is the frontier of B_2 then $S \cap f^{-1}(\delta\mathfrak{c}S) = S \cap \text{image } \rho_1$. Now let $U = (B_2 \cap f^{-1}(\delta\mathfrak{c}S)) \cup \text{image } \rho_1$.

As above, we have an imbedding $\rho_2: N_2 \times \delta_2\mathfrak{c}S \to U$ with $\delta_2 < \delta$ which agrees with ρ_1 on the overlap $(N_1 \cap N_2) \times \delta_2\mathfrak{c}S$ so that $f \circ \rho_2(x, y) = y$ and $\rho_2(x, \mathfrak{v}) = x$. In fact, letting $\delta_1 = \delta$, we can get a whole sequence of imbeddings $\rho_i: N_i \times \delta_i\mathfrak{c}S \to U$ with $\delta_i < \delta_{i-1}$ so that $f \circ \rho_i(x, y) = y$, $\rho_i(x, \mathfrak{v}) = x$ and ρ_i agrees with ρ_{i-1} on the overlap $(N_{i-1} \cap N_i) \times \delta_i\mathfrak{c}S$. Let $A_i = \delta_i\mathfrak{c}S - \mathrm{Int}(\delta_{i+1}\mathfrak{c}S) = S \times [\delta_{i+1}, \delta_i]$. Let $C_i = U \cap f^{-1}(A_i)$. Since CAT is nice, we know that $f: C_i \to A_i$ is a bundle map. But

$A_i = S \times [\delta_{i+1}, \delta_i]$, hence each C_i is also a product with an interval. More precisely, letting $U_i = U \cap f^{-1}(S \times \delta_i)$ we have an isomorphism $\varphi_i: U_i \times [\delta_{i+1}, \delta_i] \to C_i$ so that $f \circ \varphi_i(x,t) = tf(x)/\delta_i$ and $\varphi_i(x, \delta_i) = x$. Furthermore, by being careful we can also insure that $\varphi_i(\rho_j(x,y),t) = \rho_j(x,(y,t))$ for all $j \leq i$, $x \in N_j - N_{j+1}$, $y \in S$ and $t \in [\delta_{i+1}, \delta_i]$. Note we have isomorphisms $\kappa_i: U_1 \to U_i$ defined inductively by $\kappa_i = \varphi_{i-1}(\kappa_{i-1}(x), \delta_i)$. Let $N = U \cap f^{-1}(\mathfrak{v})$. We can now define an isomorphism $\varphi: U_1 \times (0,1] \to U - N$ by letting $\varphi(x,t) = \varphi_i(\kappa_i(x), \delta t)$ if $\delta_{i+1} \leq \delta t \leq \delta_i$. We can also define an imbedding $\rho: (N-p) \times \delta \mathfrak{c} S \to U$ by setting $\rho(x,(z,s)) = \rho_i(x,(z,s))$ if $x \in N_i - N_{i+1}$ and $s \leq \delta_i$ and by setting $\rho(x,(z,s)) = \varphi(\kappa_i^{-1}(\rho(x,(z,\delta_i))), s/\delta)$ if $x \in N_i - N_{i+1}$ and $s \geq \delta_i$. Note that $f\rho(x,y) = y$, $f\varphi(x,t) = tf(x)$, $\rho(x,\mathfrak{v}) = x$, $\varphi(x,1) = x$ and $\varphi(\rho(x,y),t) = \rho(x,ty)$ if $y \in S \times \delta$. It remains to show that φ extends continuously to a map $\varphi: U_1 \times [0,1] \to U$. Suppose $x = \rho(z,y) \in U_1$, then $\lim_{t\to 0} \varphi(x,t) = z$. Now suppose $x \in U_1 - \text{image } \rho$. Let z be a limit point of $\varphi(x,t)$ as $t \to 0$. Since $f\varphi(x,t) = tf(x)$ we know that $f(z) = \mathfrak{v}$. But if $z \neq p$ then we must have $x = \rho(z, f(x))$, a contradiction. So $\lim_{t\to 0} \varphi(x,t) = p$. Thus φ extends continuously.

References

[**GM**] M. Goresky and R. MacPherson, *Stratified Morse Theory*, vol. 14, Springer-Verlag, Ergebnisse Math., 1988.

[**K1**] H. King, *Thesis*, Univ. of Ca., Berkeley (1974).

[**K2**] H. King, *Topological Type of Isolated Critical Points*, Annals of Math. **107** (1978), 385–397.

[**K3**] H. King, *Topological Type in Families of Germs*, Inventiones Math. **62** (1980), 1–13.

[**K4**] H. King, *Sums of Isolated Critical Points*, preprint.

[**S**] L.Siebenmann, *Deformation of Homeomorphisms on Stratified Sets*, Comm. Math. Helv. **47** (1972), 123–163.

Department of Mathematics, University of Maryland, College Park, MD 20742

E-mail address: `hck@ frances.umd.edu`