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Resolving Singularities of Maps

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Abstract. A conjecture is given for resolution of singularities of analytic and algebraic functions. This conjecture can be proven in low dimensions. This conjecture is related to my work with Akbulut on topological characterization of real algebraic sets, and currently gives topological characterization in dimensions 4 and less. There is also a very beautiful application to the problem of factoring birational maps into blowups and their inverses.

1. How nice can you make a map?

I would like to talk about the following problem: what do maps look like? As is, this problem is much too general, so I will narrow the focus to something more specific, namely analytic maps. So the problem is: suppose $f\colon X \to Y$ is an analytic map between (real or complex) analytic spaces, then what does f look like. Suppose you wish to examine f near a point of X or Y, and in fact wish to see infinitesimal information. Imagine you had a microscope with variable power, in fact with power inversely proportional to the distance from your point (or submanifold) of interest so that the magnification became infinite at the point (or submanifold) in question. You could then actually see some infinitesimal information. The mathematical device which performs this microscopic function is blowing up. The blow up of a space X with center a manifold C is a map $\pi\colon X' \to X$ where X' is obtained from X by replacing each point of C with the directions in X which are normal to C.

I will now make the problem more precise. Given $f\colon X \to Y$, an analytic map between analytic spaces, can you perform a sequence of blowups of X, $\rho\colon X' \to X$ and a sequence of blowups of Y, $\pi\colon Y' \to Y$ so that there is an analytic lifting $g\colon X' \to Y'$ with $\pi g = f\rho$ and so that g is nice.

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Okay, so what am I up to, and what does 'nice' mean. In fact this question has arisen in two different contexts that I am aware of, both ending up with essentially the same idea of what 'nice' should mean. One context is in work of T. C. Kuo, which I have no details on, and another is in work of Akbulut and mine on topologically classifying algebraic varieties.

I will take a little time to explain this work with Akbulut, since it is dear to my heart. Suppose we have a real or complex algebraic variety X and we wish to understand its topology. Hironaka's theorem on resolution of singularities says that there is a sequence of blowups $\pi\colon X' \to X$ so that X' is a manifold and π is proper. So, for example in the complex case (with X irreducible), X is a quotient space of a manifold. But this doesn't help all that much in understanding the topology of X unless you have some idea of the collapsing which the map π does. However, if you have a good theorem on resolution of singularities of maps, you can assume that the collapsing maps are of a nice enough form that you get a good idea of the topology of X. So, for example, Akbulut and I can give a topological classification of real algebraic varieties of dimension less than or equal to four because we can do the above resolution of singularities of maps in low dimensions, see [**1**].

I mentioned resolution of singularities above, so let me state this theorem in a useful form. It was originally proven in [**3**]. This proof is extremely difficult but now there is a much more understandable proof by Bierstone-Milman.

THEOREM 1.1. *(resolution of singularities) Suppose that X is a real or complex analytic variety and $Z \subset X$ is a subvariety which contains the singular points of X, (thus $X - Z$ is an analytic manifold). Then there is a sequence of blowups $\pi\colon X' \to X$, whose centers all lie over Z, so that X' is nonsingular and so $\pi^{-1}(Z)$ is a union of divisors with normal crossings, i.e., $\pi^{-1}(Z)$ looks locally like a union of coordinate hyperplanes. If X is algebraic then all the centers can be chosen to be algebraic, so X' is an algebraic variety.*

DEFINITION 1.1. *As a shorthand, we will call a union of divisors with normal crossings a* tico*, short for 'transversely intersecting codimension one'. We sometimes also use the term* regular tico *for this since the definition of tico in* [**1**] *allows submersed manifolds also.*

DEFINITION 1.2. Tico coordinates *will mean local coordinates in which the tico is a union of coordinate hyperplanes.*

2. Submersive maps

Well, enough motivational digression, lets get on to what 'nice' should mean. One of the nicest kinds of maps I know are submersions. But we have no hope of blowing up to make g a submersion as the following example shows. Consider $f\colon \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^2$. Any blowing up of $\mathbb{C}$ is just the identity, so we are stuck with the critical point at 0. So we cannot hope to blow up to make

f a submersion, but as it will turn out, we can hope to blow up to make any analytic map a combination of submersion and monomial map.

For example, suppose that we have a proper analytic map $f: X \to Y$ between complex varieties where Y is one dimensional. By resolution of singularities we may as well assume that X and Y are manifolds. By Sard's theorem and Remmert's proper mapping theorem, the critical values of f form a zero dimensional analytic subset B, and by resolution of singularities, we may assume that $f^{-1}(B)$ is a tico. We can do no better than this. But let us see what we have.

The map f restricts to a submersion on $X - f^{-1}(B)$. In the complex case (and in the real case also if you exercise a certain amount of care) the fact that $f^{-1}(B)$ is a tico implies that near any point of $f^{-1}(B)$ you can take local coordinates so that the map f is a monomial (see Lemma 6.2.1 of [**1**]).

So this result I just proved on resolution of singularities of maps where Y has dimension one gives some hint as to what 'nice' should mean. It is some sort of combination of submersion and monomial map. I will now give two equivalent precise definitions. I will also abandon the vague word 'nice' in favor of the word 'submersive'.

Suppose we have an analytic map $f: X \to Y$. We also suppose we have ticos $\mathcal{A}$ in X and $\mathcal{B}$ in X. We also suppose that f is a tico map, in the complex case this only means that $f^{-1}(|\mathcal{B}|) \subset |\mathcal{A}|$, in the real case, see [**1**] for a definition. Note that the ticos $\mathcal{A}$ and $\mathcal{B}$ give stratifications of X and Y, i.e., they break them up into a disjoint union of submanifolds of various dimensions. The codimension k stratum of X would be those points which are in exactly k divisors of $\mathcal{A}$. In other words, in tico coordinates the point is the intersection of exactly k hyperplanes.

We will show that the following two definitions are equivalent.

Definition 2.1. *The map f is* submersive *if there are everywhere tico coordinates on X and Y so that f is a monomial map and furthermore, each connected component of a stratum of X is submersed to a stratum of Y.*

Definition 2.2. *The map f is* submersive *if the logarithmic Jacobian of f has maximal rank everywhere.*

3. The logarithmic Jacobian

Hey, where did this logarithmic Jacobian come from? Take local tico coordinates in X and Y. Take logarithms of those coordinates in X and Y which define the ticos A and B. Take the Jacobian matrix of the resulting map. It is only defined on $X - A$, but in fact it will extend to A. This is the logarithmic Jacobian.

For example, consider $f: \mathbb{C}^3 \to \mathbb{C}^3$ given by $f(x, y, z) = (x^2y^3, xy^4e^x, z + xy)$. Let the ticos in both copies of $\mathbb{C}^3$ be given by $xy = 0$. Then letting $u = \ln x$ and $v = \ln y$ our map becomes $(u, v, z) \mapsto (2u + 3v, u + 4v + e^u, z + e^{u+v})$ which has

Jacobian matrix

$$\begin{pmatrix} 2 & 3 & 0 \\ 1+x & 4 & 0 \\ xy & xy & 1 \end{pmatrix}$$

So if one deleted the affine hyperplane $x = 5/3$ the map f would be submersive.

Using Definition 2.1, we can see it is submersive when $x \neq 5/3$ by making the coordinate change $r = xe^{-.6x}$, $s = ye^{.4x}$, $t = z + xy$. Then $f(r,s,t) = (r^2s^3, rs^4, t)$, so f is a pure monomial map and it submerses each stratum of positive codimension to the codimension 2 stratum $x = y = 0$.

Note that the logarithmic Jacobian depends on the tico coordinates chosen and I do not see it as any kind of natural linear transformation, just a matrix. However, one can extract invariants of the matrix such as its rank which do not depend on the choice of tico coordinates. Later we will also find useful the ideal generated by its $k \times k$ minor determinants. The reason the rank and these ideals are well defined is that changing tico coordinates changes the logarithmic Jacobian by multiplying by the logarithmic Jacobian of the coordinate change map, which is easily seen to be nonsingular.

4. Blowing up a map to make it submersive

The work Akbulut and I did leads naturally to the following conjecture, which if proven in a reasonable technical form will imply a topological characterization of real algebraic varieties in all dimensions. The term *critical point* below means a point of X which is either a singular point of X or maps to a singular point of Y or is nonsingular and maps to a nonsingular point, but the derivative map df has rank less than the dimension of Y.

CONJECTURE 4.1. *Let $f: X \to Y$ be a proper analytic map between real or complex analytic varieties. Suppose the critical points of f are nowhere dense in X. Then there are sequences of blowups $\pi: Y' \to Y$ and $\rho: X' \to X$, ticos $\mathcal{A}$ and $\mathcal{B}$ in X' and Y', and an analytic map $g: (X', \mathcal{A}) \to (Y', \mathcal{B})$ so that $\pi \circ g = f \circ \rho$ and g is submersive. In the algebraic case (X and Y algebraic varieties, f regular) then g is regular and the blowups are algebraic.*

THEOREM 4.1. *The above conjecture is true if* $\dim Y \leq 1$ *or if* $\dim X \leq 3$.

The case $\dim Y = 0$ is trivial and the proof for $\dim Y = 1$ given above in the complex analytic case works in the real or algebraic setting also. The case $\dim X = \dim Y = 2$ is contained in the proof of Theorem 7.4.1 of [**1**]. The cases $\dim X = 3$ and $\dim Y = 2$ or 3 are considerably harder. The proof will appear elsewhere but I will outline the main ideas later in this paper.

When Akbulut and I first realized we should prove the above conjecture, we despaired of ever doing so since it is more difficult than resolution of singularities of varieties and Hironaka's proof is quite delicate, not appearing to lend itself to modification for other ends. That is why we found Bierstone's and Milman's simple proof of resolution of singularities so exciting. Finally, an ordinary mathematician can understand resolution of singularities. Moreover, since the proof uses a locally defined complexity, it can be adapted for other uses.

5. A simplicial complex associated to a tico

Suppose $(X, \mathcal{A})$ is a regular tico. We may associate to it a simplicial complex $\mathcal{K}(X, \mathcal{A})$ as follows. The vertices of $\mathcal{K}(X, \mathcal{A})$ are the nonempty sheets (i.e., divisors) of $\mathcal{A}$. There is a k-simplex with vertices $A_0, A_1, \ldots, A_k$ if the intersection $\bigcap_{i=0}^k A_i$ is nonempty. We let $|\mathcal{K}(X, \mathcal{A})|$ denote the underlying topological space of this complex.

Let $f\colon (X, \mathcal{A}) \to (Y, \mathcal{B})$ be a tico map with constant exponents. The notion of *constant exponents* is defined in [**1**], but a tico map will automatically have constant exponents if all the divisors in $\mathcal{A}$ and $\mathcal{B}$ are irreducible. We suppose also that f is *exact* which means that $f^{-1}(|\mathcal{B}|) = |\mathcal{A}|$. Then there is an induced map

$$\mathcal{K}(f)\colon |\mathcal{K}(X, \mathcal{A})| \to |\mathcal{K}(Y, \mathcal{B})|$$

which satisfies the usual functorial properties $\mathcal{K}(f \circ g) = \mathcal{K}(f) \circ \mathcal{K}(g)$ and $\mathcal{K}(id) = id$. This map is not piecewise linear, it involves radial projection which is not piecewise linear. This map is determined by the exponents of the map f, i.e., when you look at the map in local coordinates it is a monomial and the constant exponent hypothesis means the exponents of this monomial depend only on the sheets involved, not the point chosen.

Suppose $\pi\colon (X', \mathcal{A}') \to (X, \mathcal{A})$ is a tico blowup with nonempty center $C = \bigcap_{i=0}^k A_i$ for some sheets $A_i \in \mathcal{A}$. (Here *tico blowup* just means that the sheets in $\mathcal{A}'$ consist of the strict transforms of the sheets of $\mathcal{A}$ and also the exceptional divisor $\pi^{-1}(C)$.) Then the map $\mathcal{K}(\pi)$ is a homeomorphism and in fact gives a subdivision of $\mathcal{K}(X, \mathcal{A})$ as follows. Let σ be the k-simplex of $\mathcal{K}(X, \mathcal{A})$ with vertices $A_0, \ldots, A_k$. Then $\mathcal{K}(X', \mathcal{A}')$ is obtained from $\mathcal{K}(X, \mathcal{A})$ by replacing the closed star of σ with the cone on its boundary. The map $\mathcal{K}(\pi)$ takes the new vertex, the exceptional divisor, to the barycenter of σ.

For example, consider a manifold X with a tico $\mathcal{A}$ consisting of 4 sheets A_1, A_2, A_3, A_4. We are given that $A_1 \cap A_2 \cap A_3$ and $A_1 \cap A_2 \cap A_4$ are nonempty, but $A_1 \cap A_2 \cap A_3 \cap A_4$ is empty. Then $\mathcal{K}(X, \mathcal{A})$ consists of two triangles joined along the side $A_1 A_2$.

We will perform two sequences of blowups. For the first sequence we blow up $A_1 \cap A_2$, after which we blow up the intersection of the strict transform of A_3 with the exceptional divisor. For the second sequence, we first blow up $A_1 \cap A_2 \cap A_3$ after which we blow up the intersection of the strict transforms of A_1 and A_2.

Figure 1 indicates the changes to the associated simplicial complex as we perform the indicated blowups. In the first sequence, we blow up $A_1 \cap A_2$. The resulting space has an exceptional divisor, the inverse image of $A_1 \cap A_2$ which we denote A_5, the corresponding vertex in the complex being labeled 5 in the

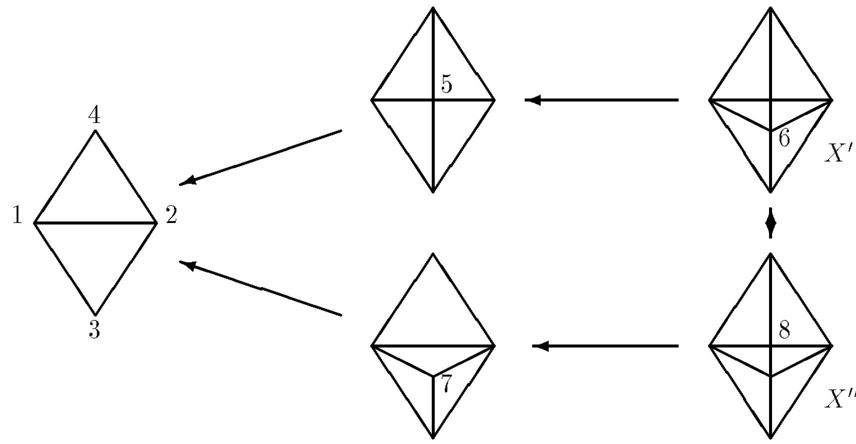

FIGURE 1. $\mathcal{K}(X,\mathcal{A})$ example

figure. We then blow up the intersection of A_5 and the strict transform of A_3, giving a new exceptional divisor A_6'. Call the resulting space X'. In X' is a tico $\mathcal{A}' = \{A_1', A_2', \dots, A_6'\}$ where A_i' is the strict transform of A_i. For the second sequence, we blow up $A_1 \cap A_2 \cap A_3$ and denote the resulting exceptional divisor A_7. We then blow up the intersection of the strict transforms of A_1 and A_2, with the resulting divisor A_8''. Call the resulting space X''. In X'' is a tico $\mathcal{A}'' = \{A_1'', A_2'', \dots, A_4'', A_7'', A_8''\}$ where A_i' is the strict transform of A_i.

In looking at the complex $\mathcal{K}(X',\mathcal{A}')$ in Figure 1, the reader may think I am cheating at first, since A_6 was supposed to appear at the barycenter of the simplex between 5 and 3, it looks in the figure as if it is 1/3 the way along this simplex. But this is right, since the the map identifying this subdivision with our original simplicial complex $\mathcal{K}(X,\mathcal{A})$ is not piecewise linear. This is the source of the apparent distortion. More precisely, A_5 is mapped to the point with projective barycentric coordinates $(1 : 1 : 0)$ which is the barycenter of A_1A_2, but A_6 is mapped to the point with projective barycentric coordinates $(1 : 1 : 1)$ since $(1,1,0) + (0,0,1) = (1,1,1)$. This point is not on the center of the line from $(0 : 0 : 1)$ to $(1 : 1 : 0)$ but is 1/3 the way along, and actually coincides with the barycenter of the triangle $A_1A_2A_3$.

I should tell the reader what projective barycentric coordinates are. A point in a simplex with projective barycentric coordinates $(a_1 : a_2 : \dots : a_n)$ has barycentric coordinates $(a_1/a, a_2/a, \dots, a_n/a)$ where $a = \sum_{i=1}^n a_i$. As we will see below, it is often convenient to describe a point's projective barycentric coordinates. This could have been avoided by defining $\mathcal{K}(X,\mathcal{A})$ differently, say taking the open cone on the $\mathcal{K}(X,\mathcal{A})$ I have defined here. Then $\mathcal{K}(f)$ would be PL, and you would not need exactness of f to define $\mathcal{K}(f)$. But I find it preferable to define things as I have here and use projective barycentric coordinates.

Note that we end up with the same complex at the end of each of the two sequences of blowups. In fact the subdivision of $\mathcal{K}(X,\mathcal{A})$ is exactly the same

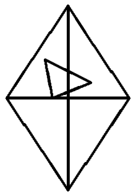

FIGURE 2. Example of map which doesn't lift

subdivision. As we shall see, this means that the resulting ticos are the same. There is an isomorphism $h\colon X' \to X''$ so that $h(A_i') = A_i''$ for $i = 1, 2, 3, 4$ and $h(A_5') = A_8''$ and $h(A_6') = A_7''$. This follows from the following discussion.

First of all, suppose $\pi\colon (X', \mathcal{A}') \to (X, \mathcal{A})$ is a sequence of tico blowups so that all centers are intersections of sheets. As we said above, the map $\mathcal{K}(\pi)\colon |\mathcal{K}(X', \mathcal{A}')| \to |\mathcal{K}(X, \mathcal{A})|$ is a homeomorphism and in fact gives a subdivision of $|\mathcal{K}(X, \mathcal{A})|$. Now suppose that $f\colon (Y, \mathcal{B}) \to (X, \mathcal{A})$ is an exact tico map with constant exponents. It is useful to know if f lifts to a map to X'. The amazing result is that f lifts to a map $g\colon Y \to X'$ with $f = \pi g$ if and only if the map $\mathcal{K}(f)$ has the the property that for each simplex σ of $\mathcal{K}(Y, \mathcal{B})$ there is a simplex δ of $\mathcal{K}(X', \mathcal{A}')$ so that $\mathcal{K}(f)(\sigma) \subset \mathcal{K}(\pi)(\delta)$. In other words, $\mathcal{K}(f)$ maps nicely into the resulting subdivision. Furthermore, if the lifting g exists then it is a tico map. For example, Figure 2 shows an example of a map which doesn't lift. The blowup is the first blowup with center $A_1 \cap A_2$ in the above example. The skewed triangle is the image of some simplex. Since it is not contained in a simplex of the subdivision, the map does not lift.

Going back to our example, we see that the map $\pi'\colon (X', \mathcal{A}') \to (X, \mathcal{A})$ lifts to a map $h\colon (X', \mathcal{A}') \to (X'', \mathcal{A}'')$ and $\pi''\colon (X'', \mathcal{A}'') \to (X, \mathcal{A})$ lifts to a map $h'\colon (X'', \mathcal{A}'') \to (X', \mathcal{A}')$. Since hh' and $h'h$ must be the identity on dense subsets (the complement of the inverse images of the centers) they give an isomorphism.

6. Factoring birational maps

This simplicial complex assigned to a tico seems useful in investigating the problem of factoring birational maps. The reason is that if f is birational and submersive and a few mild assumptions are satisfied, then $\mathcal{K}(f)$ is a homeomorphism (Theorem 7.6). What is more, there are strong conditions on this homeomorphism given by Theorem 7.7, the matrix formed by the coordinates of the images of the vertices of a simplex has determinant ± 1.

So suppose you have a birational map $f\colon X \to Y$ between complex algebraic varieties and wish to prove that it is a composition of blowup maps and their inverses, or, who knows, find obstructions to its having such a decomposition. As a first step, assuming you know the map resolution conjecture Theorem 4.1, you may as well assume that there are ticos $\mathcal{A}$ and $\mathcal{B}$ so that $f\colon (X, \mathcal{A}) \to (Y, \mathcal{B})$ is an exact submersive tico map. At that point you try to see if f is a composition of tico blowups and their inverses. But here is where the simplicial complex associated to a tico is useful. The map $\mathcal{K}(f)\colon \mathcal{K}(X,\mathcal{A}) \to \mathcal{K}(Y,\mathcal{B})$ is a homeomorphism between complexes satisfying strong conditions. A tico blowup corresponds to a certain kind of subdivision of $\mathcal{K}(X,\mathcal{A})$ or $\mathcal{K}(Y,\mathcal{B})$ by Theorem 7.4. You know when the map f lifts to a tico blowup by Theorem 7.5. So the problem reduces to finding subdivisions of $\mathcal{K}(X,\mathcal{A})$ and $\mathcal{K}(Y,\mathcal{B})$ so that $\mathcal{K}(f)$ sends vertices to vertices.

In the next section I give a proof along these lines in the dimension two case, a proof of the fact that a birational map can be factored into blowups and their inverses. I also give some partial results in dimension three. It seems to be a very interesting problem to determine whether this program can be worked out in general. Although Mori's program makes the dimension three problem not as useful as it was once thought, in higher dimensions it might be of some value. Moreover, the subdivision problem is eminently computable.

7. Proofs

In this section I give proofs or proof sketches of the results I talked about earlier. In this section I will assume some familiarity with notions defined in [**1**].

THEOREM 7.1. *The two definitions of* submersive *given in Definition* 2.1 *and Definition* 2.2 *are equivalent.*

PROOF. Suppose first that f is submersive, according to Definition 2.1 which is the definition used in [**1**]. Then an easy calculation shows that the logarithmic Jacobian has maximal rank everywhere. See the proof of Lemma 3.6.1 of [**1**].

Now suppose the logarithmic Jacobian has maximal rank everywhere. We will show that f is submersive according to Definition 2.1. Pick any point $p \in X$ and choose tico coordinates $\theta\colon (\mathbb{R}^m, 0) \to (X, p)$ and $\psi\colon (\mathbb{R}^n, 0) \to (Y, f(p))$ around p and $f(p)$. If f_i is the i-th coordinate of $\psi^{-1} f \theta$ then we have $f_i(x) = \prod_{j=1}^{\ell} x_j^{\beta_{ij}} \varphi_i(x)$ for all $i \leq k$ and $\varphi_i(0) \neq 0$. This is just what it means for f to be a tico map. Here the sheets of the tico in X near p are given by $x_i = 0$ for $i \leq \ell$ and the sheets of the tico in Y near $f(p)$ are given by $x_i = 0$ for $i \leq k$. The logarithmic Jacobian in these coordinates at the point p is:

$$\begin{pmatrix} \beta_{ij} & \partial \ln \varphi_i / \partial x_j \\ 0 & \partial f_i / \partial x_j \end{pmatrix}$$

This has rank n so the bottom right $(n-k) \times (m-\ell)$ submatrix has rank $n-k$. Consequently we may make a coordinate change in the last $m-\ell$ coordinates in X so that $f_i(x) = x_{i+m-n}$ for all $i > k$. This also shows that the stratum containing p, given locally by $x_1 = x_2 = \cdots = x_\ell = 0$, submerses to the stratum containing $f(p)$, given locally by $x_1 = x_2 = \cdots = x_k = 0$.

Let b be the rank of the upper left $k \times \ell$ matrix (β_{ij}). After reordering we may assume the first b rows of this matrix are linearly independent. Then after doing a coordinate change of the form $x_i' = x_i \prod_{j=1}^{b} \varphi_j(x)^{\gamma_{ij}}$ for $i \leq \ell$ and $x_i' = x_i$ for

$i > \ell$ we may assume that $\varphi_i(x) = 1$ for all $i \le b$. So the logarithmic Jacobian at p now looks like

$$\begin{pmatrix} \beta_{ij} & 0 & 0 \\ \beta_{ij} & \partial \ln \varphi_i/\partial x_j & \partial \ln \varphi_i/\partial x_j \\ 0 & 0 & I \end{pmatrix}$$

The middle $(k-b) \times (m - \ell - n + k)$ matrix must have rank $k - b$. After reordering the coordinates we may assume that the first $k - b$ columns of this middle matrix are linearly independent. Now do the coordinate change $x'_i = x_i$ if $i \le \ell$ or $i > \ell + k - b$ and $x'_i = \varphi_{i+b-\ell}$ if $\ell < i \le \ell + k - b$. After this coordinate change we get $\varphi_i(x) = x_{i-b+\ell}$ for all $b < i \le k$. Thus in these coordinates, f is a monomial map. Hence f is submersive according to the first definition. □

THEOREM 7.2. *The map resolution conjecture Theorem* 4.1 *is true if* X *has dimension 3.*

PROOF. An outline of my proof for the case $\dim X = 3$ is as follows. For each $k \le \dim Y$ there is a coherent sheaf $\mathcal{I}_k(f)$ of ideals of $\mathcal{O}(X)$ locally generated by the determinants of $k \times k$ minors of the logarithmic Jacobian of f. It is elementary to check that this ideal is independent of the coordinate chart chosen. Of course, f is submersive if and only if $\mathcal{I}_{\dim Y}(f) = \mathcal{O}(X)$.

The proof is in two stages. First find some sequence of blowups $\pi\colon X' \to X$ so that $\mathcal{I}_2(f \circ \pi)$ is locally principal. This is done using the Bierstone-Milman invariant of $\mathcal{I}_2$. Note it is quite easy, given a coherent sheaf of ideals, to use the Bierstone-Milman invariant to blow up and make it locally principal. This was the companion theorem to resolution of singularities in [**3**]. Its proof using the Bierstone-Milman invariant is essentially the same as their proof of the hypersurface case in [**2**] and thus is even easier than the proof of resolution of singularities since no special bases for ideals are needed. The only difference with the hypersurface case is that instead of starting out the invariant using $(g, 1)$ for g a local equation of the hypersurface, you start it out with $\{(g_1, 1), \ldots, (g_k, 1)\}$ where $g_1, \ldots, g_k$ locally generate the ideal. Everything else is the same as the hypersurface case.

So at first glance, it would seem to be no trouble to do this first step of blowing up to make $\mathcal{I}_2$ locally principal. But it is not that easy. The problem is that $\mathcal{I}_2(f \circ \pi)$ is not exactly the transform $\pi^*\mathcal{I}_2(f)$ of the ideal $\mathcal{I}_2(f)$. It is close but not quite the same. But it is close enough that the Bierstone-Milman invariant can be used if $\dim X = 3$. Usually, this invariant will not increase after blowing up a center on which it is maximal, but sometimes it does increase. This only happens in very special situations which you can deal with on a case by case basis.

Anyway, after the first step you may as well assume that $\mathcal{I}_2(f)$ is locally principal. If $\dim Y = 3$ then $\mathcal{I}_3(f)$ is automatically locally principal also since there is only one 3×3 minor. So all $\mathcal{I}_k(f)$ are locally principal and their zeroes are ticos. Consequently, there are very particular local forms for the map f. One then uses some ad hoc invariants such as those used in the proof of Theorem 7.4.1 of [**1**] to blow up X and Y some more to make f submersive. $\square$

THEOREM 7.3. *Let $f\colon (X,\mathcal{A}) \to (Y,\mathcal{B})$ be an exact tico map with constant exponents. Then there is an induced map $\mathcal{K}(f)\colon \mathcal{K}(X,\mathcal{A}) \to \mathcal{K}(Y,\mathcal{B})$ which satisfies the usual functorial properties $\mathcal{K}(f\circ g) = \mathcal{K}(f)\circ\mathcal{K}(g)$ and $\mathcal{K}(id) = id$.*

PROOF. If $A \in \mathcal{A}$ and $B \in \mathcal{B}$, let α_{AB} be the exponent of A in B for the map f. This means $f^*(\mathfrak{I}(B)) = \prod_{A\in\mathcal{A}} \mathfrak{I}(A)^{\alpha_{AB}}$. Let

$$\mathcal{B}_A = \{\, B \in \mathcal{B} \mid \alpha_{AB} > 0 \,\} = \{\, B \in \mathcal{B} \mid f(A) \subset B \,\}$$

Note $\mathcal{B}_A$ is nonempty if A is, since f is exact. Now $f(A) \subset \bigcap_{B\in\mathcal{B}_A} B$ so if $A \in \mathcal{A}$ is nonempty, then the vertices in $\mathcal{B}_A$ form a simplex σ in $\mathcal{K}(Y,\mathcal{B})$. Number the sheets in $\mathcal{B}_A$ is some way $B_1,\dots,B_\ell$. Let α_i be the exponent of A in B_i. Let $\mathcal{K}(f)(A)$ be the point in σ with projective barycentric coordinates $(\alpha_1 : \alpha_2 : \cdots : \alpha_\ell)$.

It might seem natural at this point to extend linearly to obtain a PL map. However, if one did so the composition rule would not hold. What we do instead is the following. Take a point p in a simplex δ of $\mathcal{K}(X,\mathcal{A})$. Denote the vertices of δ as $A_1,\dots,A_k$ and suppose p has barycentric coordinates $(t_0,\dots,t_k)$. Let

$$\mathcal{B}_\delta = \bigcup_{i=1}^{k} \mathcal{B}_{A_i} = \{\, B \in \mathcal{B} \mid f(\bigcap_{i=1}^{k} A_i) \subset B \,\}$$

Let σ be the simplex of $\mathcal{K}(Y,\mathcal{B})$ with vertices $\mathcal{B}_\delta$. Number the sheets of $\mathcal{B}_\delta$ in some way $B_1,\dots,B_\ell$. Let α_{ji} be the exponent of A_j in B_i. Then $\mathcal{K}(f)(p)$ is the point of σ with projective barycentric coordinates $(\beta_1 : \cdots : \beta_\ell)$ where $\beta_i = \sum_{j=1}^{k} t_j\alpha_{ji}$. It is elementary to check that the functorial properties are satisfied.

Note that the image of a simplex is the convex hull of the images of its vertices. So on that level it looks like a PL map. It is just that the map $\mathcal{K}(f)$ itself is not PL, because radial projection is not a PL map. $\square$

There is a more concrete construction of $\mathcal{K}(X,\mathcal{A})$ and $\mathcal{K}(f)$ which the reader may find helpful. We assume $\mathcal{A}$ is finite. Suppose there are n nonempty sheets in $\mathcal{A}$. Then we can realize $\mathcal{K}(X,\mathcal{A})$ as a subset of $\mathbb{R}^n$ as follows. Order the nonempty sheets of $\mathcal{A}$ as $A_1,\dots,A_n$. Let e_i be the standard basis vector in $\mathbb{R}^n$. Then $\{\, x \in \mathbb{R}^n \mid x_i \geq 0 \ \text{ and } \ \sum_{i=1}^{n} x_i = 1 \,\}$ is a standard $(n-1)$-simplex, the convex hull of $e_1,\dots,e_n$. Then $\mathcal{K}(X,\mathcal{A})$ is the union of the of the faces of this simplex corresponding to nonempty intersections of the A_i's. In other words, $|\mathcal{K}(X,\mathcal{A})|$ is the union of convex hulls of $e_{i_1},\dots,e_{i_k}$ where $\bigcap_{j=1}^{k} A_{i_j}$ is nonempty. If $\mathcal{B}$ has m nonempty sheets we have an $m\times n$ matrix L of exponents of the map f. Then $\mathcal{K}(f)$ is the composition of L and radial projection to the affine hyperplane $\sum_{i=1}^{m} y_i = 1$.

Let us now see the effect of a tico blowup on this simplicial complex. It is not hard to see the effect in general, but we will restrict our attention to the case where the center of the blowup is an intersection of sheets. In this case, the effect is to subdivide $\mathcal{K}(X,\mathcal{A})$ in a certain way. In particular:

THEOREM 7.4. *Let $\pi\colon (X',\mathcal{A}') \to (X,\mathcal{A})$ be a tico blowup with nonempty center $C = \bigcap_{i=0}^{k} A_i$ for some sheets $A_i \in \mathcal{A}$. Then $\mathcal{K}(X',\mathcal{A}')$ is isomorphic to a subdivision of the simplicial complex $\mathcal{K}(X,\mathcal{A})$. This subdivision is as follows. Let σ be the k-simplex of $\mathcal{K}(X,\mathcal{A})$ with vertices $A_0,\dots,A_k$. Then $\mathcal{K}(X',\mathcal{A}')$ is obtained from $\mathcal{K}(X,\mathcal{A})$ by replacing the closed star of σ with the cone on its boundary.*

PROOF. We may suppose that $k > 0$, otherwise $(X,\mathcal{A})$ and $(X',\mathcal{A}')$ are isomorphic and the result is trivial. Consider the map $\mathcal{K}(\pi)\colon |\mathcal{K}(X',\mathcal{A}')| \to |\mathcal{K}(X,\mathcal{A})|$. The exponents of the tico map π are given in Lemma 3.3.4 of [**1**]. If $A \in \mathcal{A}$ is a nonempty sheet, then $\mathcal{K}(\pi)$ sends the vertex the strict transform of A to the vertex A. Also, $\mathcal{K}(\pi)$ sends the sheet $\pi^{-1}(C)$ to the barycenter of σ. It is elementary to check that $\mathcal{K}(\pi)$ is a homeomorphism and performs the indicated subdivision. □

We have the following criterion for lifting maps.

THEOREM 7.5. *Let $f\colon (X,\mathcal{A}) \to (Y,\mathcal{B})$ be an exact tico map with constant exponents and let $\pi\colon (Y',\mathcal{B}') \to (Y,\mathcal{B})$ be a tico blowup whose center C is an intersection of sheets. Then we may lift f to a tico map $g\colon (X,\mathcal{A}) \to (Y',\mathcal{B}')$ with $f = \pi\circ g$ if and only if for each simplex σ of $\mathcal{K}(X,\mathcal{A})$, the image $\mathcal{K}(\pi)^{-1}\circ\mathcal{K}(f)(\sigma)$ is contained in a simplex of $\mathcal{K}(Y',\mathcal{B}')$.*

PROOF. If g exists, then $\mathcal{K}(\pi)^{-1}\circ\mathcal{K}(f) = \mathcal{K}(g)$ which takes simplices into simplices. So this implication is trivial.

Now suppose that $\mathcal{K}(\pi)^{-1}\circ\mathcal{K}(f)$ takes simplices into simplices. Let $B_0,\dots,B_k$ be the sheets of $\mathcal{B}$ so that $C = \bigcap_{i=0}^{k} B_i$. Take a point $q \in X$ so that there is no local lifting of f near q. Let $A_0,\dots,A_\ell$ be the sheets of $\mathcal{A}$ which contain q. Let σ be the ℓ-simplex of $\mathcal{K}(X,\mathcal{A})$ with vertices $A_0,\dots,A_\ell$. Let δ be the smallest simplex of $\mathcal{K}(Y,\mathcal{B})$ which contains $\mathcal{K}(f)(\sigma)$. Let δ have dimension m. Note that we must have $f(q) \in C$ since f does not lift near q, so for each B_i we must have the exponent of A_j in B_i is greater than zero for some j. Consequently, each B_i must be a vertex of δ. We may thus name the vertices of δ as $B_0,\dots,B_m$. The map $\mathcal{K}(\pi)$ subdivides δ as indicated in Theorem 7.4. There are $k+1$ new m-simplices in the subdivision, the i-th one consists of all points of δ whose i-th barycentric coordinate is less than or equal to all the j-th barycentric coordinates for $0 \le j \le k$. Consequently, after reordering we may assume that for each point in $\mathcal{K}(f)(\sigma)$, its 0-th barycentric coordinate is less than or equal to all the j-th barycentric coordinates for $0 \le j \le k$. In particular, looking at $\mathcal{K}(f)(A_i)$ we see that if α_{ij} is the exponent of A_i in B_j, then $\alpha_{i0} \le \alpha_{ij}$ for all $0 \le j \le k$. Now $\Im(C) = \sum_{i=0}^{k} \Im(B_i)$ so locally we have

$$f^*(\Im(C)) = \sum_{j=0}^{k} \prod_{i=0}^{\ell} \Im(A_i)^{\alpha_{ij}} = \prod_{i=0}^{\ell} \Im(A_i)^{\alpha_{i0}}$$

so $f^*(\Im(C))$ is locally principal and f lifts near q, a contradiction. □

LEMMA 7.1. *Let $f\colon (X,\mathcal{A}) \to (Y,\mathcal{B})$ be an exact tico map with constant exponents. Suppose the map $\mathcal{K}(f)$ takes the interior of a simplex δ into the interior of a simplex σ. (Thus σ is the smallest simplex of $\mathcal{K}(Y,\mathcal{B})$ which contains $\mathcal{K}(f)(\delta)$.) Suppose δ has vertices $A_1, \ldots, A_k$ and σ has vertices $B_1, \ldots, B_\ell$. Pick some point $p \in \bigcap_{i=1}^{k} A_i$ which is in no other sheets of $\mathcal{A}$. Then $f(p)$ is in the sheets B_i and in no other sheets of $\mathcal{B}$.*

PROOF. Suppose $f(p) \in B$, for some sheet B in $\mathcal{B}$. We know that $f^{-1}(B)$ is a union of sheets by Lemma 3.2.1 of [**1**] Since $p \in f^{-1}(B)$ and p is only in the A_i's we know that one of the sheets A_i is in $f^{-1}(B)$. But then Lemma 3.2.2 of [**1**] says A_i must have a positive exponent with B, so $\mathcal{K}(f)(A_i)$ has nontrivial B coordinate and B must be a vertex of σ. □

LEMMA 7.2. *Let $f\colon (X,\mathcal{A}) \to (Y,\mathcal{B})$ be a submersive exact tico map with constant exponents between complex ticos and suppose that f is a proper analytic map. Suppose the map $\mathcal{K}(f)$ takes the interior of a simplex δ into the interior of a simplex σ. Let $A_0, \ldots, A_n$ be the vertices of δ and $B_0, \ldots, B_m$ be the vertices of σ. Then $f(\bigcap_{i=0}^{n} A_i)$ is a union of connected components of $\bigcap_{i=0}^{m} B_i$. In the algebraic case, $f(\bigcap_{i=0}^{n} A_i)$ is a union of irreducible components of $\bigcap_{i=0}^{m} B_i$.*

PROOF. Since f is proper we know that $f(\bigcap_{i=0}^{n} A_i)$ is a subvariety of Y. Since $\mathcal{K}(f)$ takes the interior of δ to the interior of σ we know that for each $i \le m$ there is a $j \le n$ so that the exponent of A_j in B_i is nonzero. Consequently, $f(A_j) \subset B_i$. But then $f(\bigcap_{i=0}^{n} A_i) \subset \bigcap_{i=0}^{m} B_i$. So we only need show that $f(\bigcap_{i=0}^{n} A_i)$ is an open subset of $\bigcap_{i=0}^{m} B_i$. This follows from properness and the fact that f submerses strata to strata. □

Now let us see what happens to a submersive tico map with constant exponents which is also a proper birational equivalence.

THEOREM 7.6. *Let $f\colon (X,\mathcal{A}) \to (Y,\mathcal{B})$ be an exact tico map with constant exponents between complex ticos and suppose that f is proper and a birational equivalence. Suppose also that Y is irreducible and any intersection of sheets of $\mathcal{B}$ is either empty or irreducible. Then the map $\mathcal{K}(f)$ is onto. If, in addition, f is submersive then $\mathcal{K}(f)$ is a homeomorphism.*

PROOF. Pick an algebraic subvariety $Z \subset Y$ and a rational function $g\colon Y - Z \to X$ so that $f \circ g$ is the identity on $Y - Z$.

First we see that $\mathcal{K}(f)$ is onto. Take any simplex of $\mathcal{K}(Y,\mathcal{B})$. Denote its vertices $B_1,\dots,B_k$. Take some point c in the interior of this simplex with rational barycentric coordinates. Let its barycentric coordinates coordinates be $(c_1/d,\dots,c_k/d)$ for some positive integers c_i and d.

Take some point $p\in\bigcap_{i=1}^k B_i$ so that p is in no sheet of $\mathcal{B}$ except for the B_i's. Take analytic tico coordinates $\theta\colon (U,0)\to(Y,p)$ around p where U is an open subset of $\mathbb{C}^n$ so that $\theta^{-1}(B_i)=\mathbb{C}^n_i\cap U$. For each $z\in\mathbb{C}^n$ consider the analytic curve

$$\alpha_z(t)=\theta(t^{c_1}+z_1t^{2d},t^{c_2}+z_2t^{2d},\dots,t^{c_k}+z_kt^{2d},z_{k+1}t^{2d},\dots,z_nt^{2d})$$

Note that the union of the images of these curves is open and hence for some z near 0 and $\epsilon>0$, we have $\alpha_z(t)\notin Z$ for all $t\in\mathbb{C}$ with $0<|t|<\epsilon$. Then by properness of f, there is a limit point q of $g\alpha_z(t)$ as $t\to 0$. But then $q\cup g\alpha_z(\epsilon\mathbb{C}-0)$ is an analytic curve in X. (It is an irreducible component at q of the analytic set $f^{-1}(\alpha_z(\epsilon\mathbb{C}))=g\alpha_z(\epsilon\mathbb{C}-0)\cup f^{-1}(p)$.)

Denote by $A_1,\dots,A_\ell$ the sheets of $\mathcal{A}$ which contain q. Pick any analytic tico coordinates $\psi\colon (V,0)\to(X,q)$ in a neighborhood of q where V is open in $\mathbb{C}^n$ so that $\psi^{-1}(A_i)=\mathbb{C}^n_i$ for all $i\le\ell$. Since f is a tico map, we know that if f_i is the i-th coordinate of $\theta^{-1}f\psi$ then $f_i(x)=\prod_{j=1}^{\ell}x_j^{\beta_{ij}}\varphi_i(x)$ for all $i\le k$. Here β_{ij} are some nonnegative integers and $\varphi_i(0)\ne 0$.

Take an analytic parameterization of the analytic curve $\psi^{-1}(q\cup g\alpha_z(\epsilon\mathbb{C}-0))$, say $\gamma(s)=(\gamma_1(s),\dots,\gamma_m(s))$ for analytic γ_i's. Since α_z does not intersect $|\mathcal{B}|$ near p except at p and since $f^{-1}(|\mathcal{B}|)=|\mathcal{A}|$, we know for each $i\le\ell$ that γ_i is not identically 0. So let e_i be the order of γ_i for $i\le\ell$ and let $\gamma_i'(s)=s^{-e_i}\gamma_i(s)$. Note $\gamma_j'(0)\ne 0$. Now $\theta^{-1}f\psi\gamma$ is another parameterization of the curve $\theta^{-1}\alpha_z$. For $i\le k$, the i-th coordinate of $\theta^{-1}f\psi\gamma(s)$ is

$$\left(s^{\sum_{j=1}^{\ell}\beta_{ij}e_j}\right)\prod_{j=1}^{\ell}\gamma_j'(s)^{\beta_{ij}}\varphi_i(\gamma(s))$$

Consequently, comparing the relative orders of the first k coordinates in the two parameterizations, we see that there is a rational number r so that for each $i\le k$ we have $\sum_{j=1}^{\ell}\beta_{ij}e_j=rc_i$. But this means exactly that if we take the simplex in $\mathcal{K}(X,\mathcal{A})$ with vertices $A_1,\dots,A_\ell$ and take the point in it with projective barycentric coordinates $(e_1:\dots:e_\ell)$ and apply $\mathcal{K}(f)$, then we get the point c. Consequently, $\mathcal{K}(f)$ is onto since the points with rational barycentric coordinates are dense in $\mathcal{K}(Y,\mathcal{B})$.

Now we will show that $\mathcal{K}(f)$ is one to one if f is submersive. Suppose the images of two different simplices intersect. Then it is easy to see that they must intersect in some point c which has rational barycentric coordinates. This point c then has (at least) two different inverse images e and e' which also have rational barycentric coordinates. Let e be in the interior of a simplex with vertices $A_1,\dots,A_\ell$ and let e' be in the interior of a simplex with vertices $A_1',\dots,A_{\ell'}'$. Let c be in the interior of a simplex with vertices $B_1,\dots,B_k$.

Pick any point $p \in \bigcap_{i=1}^{k} B_i$ so that p is in no other sheets of $\mathcal{B}$. By Lemma 7.2, we know that $f(\bigcap_{i=1}^{\ell} A_i) = \bigcap_{i=1}^{k} B_i$ and $f(\bigcap_{i=1}^{\ell'} A_i') = \bigcap_{i=1}^{k} B_i$. So we may pick $q \in \bigcap_{i=1}^{\ell} A_i$ and $q' \in \bigcap_{i=1}^{\ell'} A_i'$ so that $f(q) = f(q') = p$.

Pick a curve α_z as we did above. I claim we may pick curves β and β' near q and q' so that $f \circ \beta = f \circ \beta' = \alpha_z$. This then is a contradiction since both curves β and β' must be $g\alpha_z$, but they are distinct (even if $q = q'$).

To see the claim, I will construct β, the curve β' is constructed similarly. By the complex version of Lemma 3.6.2 of [**1**] we may find analytic tico coordinates $\psi \colon V \to X$ in a neighborhood of q where V is open in $\mathbb{C}^n$ so that $\psi^{-1}(A_i) = \mathbb{C}^n_i$ for all $i \leq \ell$ and if f_i is the i-th coordinate of $\theta^{-1} f \psi$ then $f_i(x) = x_i$ for all $i > k$ and $f_i(x) = \prod_{j=1}^{k} x_j^{\beta_{ij}}$ for all $i \leq k$. Here β_{ij} are some nonnegative integers. Note that the i-th coordinate of $\psi^{-1}(q)$ is 0 for $i \leq \ell$ and $i > k$ but is nonzero for all other i's. Choose positive integers e_i so that e has projective barycentric coordinates $(e_1 : \ldots : e_\ell)$. Our curve β will be of the form

$$\beta(s) = \psi(s^{e_1}\gamma_1(s), \ldots, s^{e_\ell}\gamma_\ell(s), \gamma_{\ell+1}(s), \ldots, \gamma_n(s))$$

where $\gamma_i(0) \neq 0$ for all $i \leq k$. Since $\mathcal{K}(f)(e) = c$ we know there is a rational number r so that $\sum_{j=1}^{\ell} \beta_{ij} e_j = rc_i$ for each $i \leq k$. In fact, for convenience, choose the e_i so that r is an integer. So we want to solve the equation $\theta^{-1} f \beta(s) = \theta^{-1}\alpha_z(s^r)$. This becomes the equations

$$\begin{array}{rcll} \prod_{j=1}^{k} \gamma_j(s)^{\beta_{ij}} & = & 1 + z_i s^{2rd - rc_i} & \text{for } i \leq k \\ \gamma_i(s) & = & z_i s^{2rd} & \text{for } i > k \end{array}$$

Since the matrix (β_{ij}) is onto there are rational numbers κ_{ja} so that $\sum_{j=1}^{k} \beta_{ij}\kappa_{ja}$ is the Kronecker delta δ_{ia}. We may then let

$$\gamma_j(s) = \begin{cases} \prod_{i=1}^{k} (1 + z_i s^{2rd - rc_i})^{\kappa_{ji}} & \text{for } j \leq k \\ z_j s^{2rd} & \text{for } j > k \end{cases}$$

and we have found our curve β. $\square$

THEOREM 7.7. *Let $f \colon (X, \mathcal{A}) \to (Y, \mathcal{B})$ be a submersive exact tico map with constant exponents between complex ticos and suppose that f is proper and a birational equivalence. Let σ be an $\ell - 1$ simplex in $\mathcal{K}(X, \mathcal{A})$ and let δ be the smallest simplex of $\mathcal{K}(Y, \mathcal{B})$ which contains $\mathcal{K}(f)(\sigma)$. Let δ have dimension $k - 1$. We have a $k \times \ell$ matrix T formed from the exponents of the vertices of σ in the vertices of δ. (So the columns of T give projective barycentric coordinates for the images of the vertices of σ.)*

Then $T^{-1}(\mathbb{Z}^k) \subset \mathbb{Z}^\ell$, the inverse image of the integer lattice is contained in the integer lattice. Moreover, T is a submatrix of a $k \times k$ nonnegative integer matrix T' with determinant ± 1. Consequently each column of T has greatest common divisor 1.

PROOF. Let the vertices of σ be $A_1, \ldots, A_\ell$ and let $B_1, \ldots, B_k$ be the vertices δ. Pick some point $q \in \bigcap_{i=1}^k A_i$ which is in no other sheets of $\mathcal{A}$. Then by Theorem 7.1, $f(q)$ is in the sheets B_i and in no other sheets of $\mathcal{B}$.

Take analytic tico coordinates $\theta\colon (U,0) \to (Y, f(q))$ around $f(q)$ where U is an open subset of $\mathbb{C}^n$ so that $\theta^{-1}(B_i) = \mathbb{C}^n_i \cap U$. By the complex version of Lemma 3.6.2 of [**1**] we may find analytic tico coordinates $\psi\colon V \to X$ in a neighborhood of q where V is open in $\mathbb{C}^n$ so that $\psi^{-1}(A_i) = \mathbb{C}^n_i$ for all $i \leq \ell$ and if f_i is the i-th coordinate of $\theta^{-1} f \psi$ then $f_i(x) = x_i$ for all $i > k$ and $f_i(x) = \prod_{j=1}^k x_j^{\beta_{ij}}$ for all $i \leq k$. Here β_{ij} are some nonnegative integers. Note that the i-th coordinate of $\psi^{-1}(q)$ is 0 for $i \leq \ell$ and $i > k$ but is nonzero for all other i's.

Note that the ij-th entry of T is β_{ij}. Suppose that $Tc \in \mathbb{Z}^k$ for some $c \notin \mathbb{Z}^\ell$. Then for any x near $\psi^{-1}(q)$ in V, we have $f\psi(x) = f\psi(x')$ where $x'_i = x_i$ for $i > \ell$ and $x'_i = e^{2\pi c_i \sqrt{-1}} x_i$ for $i \leq \ell$. This is a contradiction to birationality of f, since f must be locally generically one to one. So we know that $T^{-1}(\mathbb{Z}^k) \subset \mathbb{Z}^\ell$.

If $\ell = k$ then this implies T has determinant ± 1. Since $\mathcal{K}(f)$ is a homeomorphism, we know that σ is a face of a $k-1$ simplex σ' so that $\mathcal{K}(f)(\sigma') \subset \delta$. The matrix of exponents T' we associate to σ' has determinant ± 1 and its first ℓ columns are exactly T. $\square$

THEOREM 7.8. *Let $f\colon (X, \mathcal{A}) \to (Y, \mathcal{B})$ be a submersive tico map satisfying the conditions of Theorem* 7.6. *Suppose also that $\mathcal{K}(f)$ takes vertices to vertices. Then f has an entire rational inverse.*

PROOF. Let the sheets of $\mathcal{A}$ be $A_1, \ldots A_k$. Then we may write the sheets as $\mathcal{B}$ as $B_1, \ldots B_k$ where $\mathcal{K}(f)(A_i) = B_i$. By Lemma 7.2 we know that $f(A_i) = B_i$. By Lemma 3.2.2 of [**1**] we also know $A_i = f^{-1}(B_i)$. Since the exponent of A_i in B_j is the Kronecker delta, the local form of the submersive tico map is by definition the identity, so f is locally an analytic isomorphism. But then since it is birational, it must be globally an isomorphism. $\square$

Although Lemma 7.3 and Lemma 7.4 below ask for the hypotheses of Theorem 7.6, in reality they require less, namely that $\mathcal{K}(f)$ be a homeomorphism and that the conclusion of Theorem 7.7 hold, more particularly, Lemma 7.3 requires that if $\mathcal{K}(f)$ maps a 1-simplex into a 1-simplex then the 2×2 matrix of exponents have determinant ± 1 and Lemma 7.4 requires that if $\mathcal{K}(f)$ maps a 2-simplex into a 2-simplex then the 3×3 matrix of exponents have determinant ± 1.

This brings to mind the following question. Suppose $(Y, \mathcal{B})$ is such that all intersections of sheets are irreducible and suppose we have a subdivision of $\mathcal{K}(Y, \mathcal{B})$ satisfying the conclusions of Theorem 7.7, i.e., we have choices of integer projective barycentric coordinates of all vertices so that all $k \times k$ matrices formed from vertices of $k-1$ simplices in a $k-1$ simplex of $\mathcal{K}(Y, \mathcal{B})$ will have determinant ± 1. Then is there an $(X, \mathcal{A})$ and a birational submersive tico map $f\colon (X, \mathcal{A}) \to (Y, \mathcal{B})$ giving this subdivision? This is true in dimensions less than or equal to 2.

LEMMA 7.3. *Let $f\colon (X,\mathcal{A}) \to (Y,\mathcal{B})$ be a tico map satisfying the conditions of Theorem 7.6. Let σ be a 1-simplex of $\mathcal{K}(Y,\mathcal{B})$ and suppose that $\mathcal{K}(f)$ takes some vertex to the interior of σ. Then $\mathcal{K}(f)$ takes some vertex to the barycenter of σ.*

PROOF. Let σ have vertices B_1 and B_2. Let c denote the barycenter of σ. Blow up the point $B_1 \cap B_2$ in Y obtaining a tico blowup $\pi\colon (Y',\mathcal{B}') \to (Y,\mathcal{B})$. By Proposition 3.3.9 of [**1**] we know that after some sequence of tico blowups $\rho\colon (X,\mathcal{A}') \to (X,\mathcal{A})$ we have a lifting $g\colon X' \to Y'$ so $\pi g = f\rho$. Moreover, the proof of Proposition 3.3.9 shows that the centers of all blowups are intersections of two sheets. We will now show that this sequence of blowups is not necessary, i.e., we could take $\rho =$ identity.

Take a ρ which has a smallest number of blowups. Suppose that some blowing up is necessary. Consider the last blowup in ρ. So $\rho = \rho''\circ\rho'$ where $\rho' : (X',\mathcal{A}') \to (X'',\mathcal{A}'')$ and $\rho''\colon (X'',\mathcal{A}'') \to (X,\mathcal{A})$ and ρ' blows up the intersection of two sheets A_1 and A_2 of $\mathcal{A}''$. Let $d \in |\mathcal{K}(X'',\mathcal{A}'')|$ be the barycenter of the simplex with vertices A_1 and A_2. By minimality of ρ we know that $f\circ\rho''$ does not lift, but $f\circ\rho$ does lift. Hence by Theorem 7.4 and Theorem 7.5 we know that $\mathcal{K}(f\circ\rho'')(d) = c$. Let $\mathcal{K}(f\circ\rho'')(A_i)$ have projective barycentric coordinates $(\beta_{1i} : \beta_{2i})$, in other words β_{ji} are the exponents of A_i in B_j for $f\circ\rho''$. By Theorem 7.7, we know that $\beta_{11}\beta_{22} - \beta_{12}\beta_{21} = \pm 1$. On the other hand, $\mathcal{K}(f\circ\rho'')(d) = c$. But $\mathcal{K}(f\circ\rho'')(d)$ has projective barycentric coordinates $(\beta_{11}+\beta_{12} : \beta_{21}+\beta_{22})$ so $\beta_{11}+\beta_{12} = \beta_{21}+\beta_{22}$. But then by Theorem 7.7, we have $\beta_{11}+\beta_{12} = \beta_{21}+\beta_{22} = 1$. So the only two possibilities for the integers β_{ji} are $\beta_{11} = \beta_{22} = 1$ and $\beta_{12} = \beta_{21} = 0$ or $\beta_{11} = \beta_{22} = 0$ and $\beta_{12} = \beta_{21} = 1$. In other words, the vertices A_i get mapped to the vertices B_j. This together with the fact that $\mathcal{K}(f\circ\rho'')$ is a homeomorphism implies that no vertex of $\mathcal{K}(X'',\mathcal{A}'')$ maps to the interior of the simplex B_1B_2. In particular, no vertex of $\mathcal{K}(X,\mathcal{A})$ is mapped to the interior so we have a contradiction. Consequently, the minimal ρ is just the identity which is what we wanted.

So the map f lifts. Take a simplex δ so that $c \in \mathcal{K}(f)(\delta)$. Then Theorem 7.4 and Theorem 7.5 imply that the image of δ lies on one side or the other of c. In particular, some vertex of δ must map to c. $\square$

Let us use these facts to show that a proper, birational map between complex surfaces is a composition of blowings up and blowings down. So let $f\colon X \to Y$ be a proper birational map between complex surfaces. We may as well assume that Y is irreducible. By Theorem 4.1, after blowing up X and Y we may as well assume there are algebraic ticos $\mathcal{A}$ in X and $\mathcal{B}$ in Y so that f is a submersive tico map. After blowing up irreducible components of intersections of sheets in Y, we may also assume that intersections of sheets in $\mathcal{A}$ and $\mathcal{B}$ are all irreducible. Lemma 3.6.3 of [**1**] shows that this extra blowing up does not destroy submersiveness. Then we may apply Theorem 7.6 to see that $\mathcal{K}(f)$ gives a homeomorphism from $|\mathcal{K}(X,\mathcal{A})|$ to $|\mathcal{K}(Y,\mathcal{B})|$.

Since X and Y are surfaces, all threefold intersections of sheets are empty, so $\mathcal{K}(X,\mathcal{A})$ and $\mathcal{K}(Y,\mathcal{B})$ both have dimensions ≤ 1. Suppose $\mathcal{K}(f)$ maps some vertex of $\mathcal{K}(X,\mathcal{A})$ to the interior of a 1-simplex of $\mathcal{K}(Y,\mathcal{B})$. Let this 1-simplex have vertices B_1 and B_2. By Lemma 7.3, some vertex of $\mathcal{K}(X,\mathcal{A})$ is mapped to the barycenter of σ. Hence every simplex which maps to σ must map to one side or the other of the barycenter. Blow up the point $B_1 \cap B_2$ in Y obtaining a tico blowup $\pi\colon (Y',\mathcal{B}') \to (Y,\mathcal{B})$. By Theorem 7.5, we have a lifting $g\colon X \to Y'$ so $\pi g = f$. We continue to blow up Y as long as there are vertices of $\mathcal{K}(X,\mathcal{A})$ which are not mapped to vertices. Eventually this process must stop since after each step there is one more vertex of $\mathcal{K}(X,\mathcal{A})$ which is mapped to a vertex. At the end we have a lifting $g'\colon (X,\mathcal{A}) \to (Y'',\mathcal{B}'')$ and $\mathcal{K}(g')$ takes vertices to vertices. By Lemma 3.6.3 of [**1**] and the following Theorem 7.8 then g' is an algebraic isomorphism. In other words, a proper submersive exact f is in fact a blowup map, if intersections of sheets are irreducible.

The next question is what happens in dimension 3. We can try the same idea there. Again we may assume that f is submersive. However, it turns out that sometimes we need to do the blowups ρ of X so a proof along these lines must be more complicated. However, the above argument takes care of the one dimensional parts of $\mathcal{K}(X,\mathcal{A})$ and we can direct our full attention to the 2-simplices. We then have the following result.

LEMMA 7.4. *Let $f\colon (X,\mathcal{A}) \to (Y,\mathcal{B})$ be a tico map satisfying the conditions of Theorem* 7.6. *Let σ be a 2-simplex of $\mathcal{K}(Y,\mathcal{B})$ with vertices B_1, B_2, B_3. Let A_1, A_2 and A_3 be the unique sheets of $\mathcal{A}$ so that $\mathcal{K}(f)(A_i) = B_i$, $i = 1,2,3$. Then exactly one of the following possibilities holds.*

(i) *Some 2-simplex of $\mathcal{K}(X,\mathcal{A})$ maps onto σ. Note that this simplex must be $A_1A_2A_3$.*
(ii) *There is a sheet A_4 of $\mathcal{A}$ so that $\mathcal{K}(f)(A_4)$ is the barycenter of σ.*
(iii) *After perhaps reordering the B_i's, there is a sheet A_5 in $\mathcal{A}$ so that $\mathcal{K}(f)(A_5)$ is the barycenter of B_2 and B_3 and so A_1 and A_5 form a simplex in $\mathcal{K}(X,\mathcal{A})$.*

PROOF. Suppose that neither the first nor the second possibility happens. First, we may assume that some sheet in $\mathcal{K}(X,\mathcal{A})$ maps to the interior of one of the edges of σ. For suppose it does not. Then A_2A_3 is a simplex of $\mathcal{K}(X,\mathcal{A})$ and so $A_3 \cap A_2$ is nonempty. Let $\rho\colon (X',\mathcal{A}') \to (X,\mathcal{A})$ be the blow up with center $A_3 \cap A_2$. After the blowup, we still have no sheets mapping to the barycenter of σ, but we do have a sheet mapping to the interior of B_3B_2 so by the following argument, the last possibility (iii) holds. So A_1', the strict transform of A_1, and $A_5 = \rho^{-1}(A_2 \cap A_3)$ form a simplex in $\mathcal{K}(X',\mathcal{A}')$. But this means that $A_1 \cap A_2 \cap A_3$ is nonempty which is a contradiction since then there would be a simplex $A_1A_2A_3$ which $\mathcal{K}(f)$ would map onto σ and the first possibility (i) would hold.

So we may assume, after reordering, that $\mathcal{K}(f)$ maps some sheet to the interior of the edge B_2B_3. By Lemma 7.3, there is a sheet A_6 of $\mathcal{A}$ so that $\mathcal{K}(f)(A_6)$ is the barycenter of B_2B_3. Let $\pi\colon (Y',\mathcal{B}') \to (Y,\mathcal{B})$ be the tico blowup with center $B_2 \cap B_3$. By Proposition 3.3.9 of [**1**], there is a sequence of tico blowups $\rho\colon (X',\mathcal{A}') \to (X,\mathcal{A})$ and a tico map $f'\colon (X',\mathcal{A}') \to (Y',\mathcal{B}')$ so that $\pi f' = f\rho$. Also each center of each blowup in ρ is the intersection of two sheets. Now there is a 1-simplex σ'' in $\mathcal{K}(Y',\mathcal{B}')$ which $\mathcal{K}(\pi)$ maps to the line from B_1 to the barycenter of the edge B_2B_3. Note $\mathcal{K}(\pi)$ takes the barycenter of σ'' to the barycenter of σ.

Suppose that some vertex of $\mathcal{K}(X',\mathcal{A}')$ is mapped by $\mathcal{K}(f')$ to the interior of σ''. Then by Lemma 7.3, there is a sheet $A \in \mathcal{A}'$ so that $\mathcal{K}(f')(A)$ is the barycenter of σ''. Then $\mathcal{K}(f)(\mathcal{K}(\rho)(A))$ is the barycenter of σ so $\mathcal{K}(\rho)(A)$ cannot be a vertex of $\mathcal{K}(X,\mathcal{A})$. So we may factor ρ as the composition $\rho = \rho'' \circ \rho' \circ \rho'''$ where $\rho'''\colon (X',\mathcal{A}') \to (X''',\mathcal{A}''')$ and $\rho'\colon (X''',\mathcal{A}''') \to (X'',\mathcal{A}'')$ and $\rho''\colon (X'',\mathcal{A}'') \to (X,\mathcal{A})$ and ρ' is a single blowup with center C and $\mathcal{K}(\rho''')(A)$ is the vertex $A' = \rho'^{-1}(C)$.

Let $C = A_1'' \cap A_2''$ for sheets $A_i'' \in \mathcal{A}''$. Let β_{ji} be the exponent of A_i'' in B_j for $f \circ \rho''$. So $\mathcal{K}(f \circ \rho'')(A_i'')$ has projective barycentric coordinates $(\beta_{1i} : \beta_{2i} : \beta_{3i})$. Also $\beta_{j1} + \beta_{j2}$ is the exponent of A' in B_j for the map $f \circ \rho'' \circ \rho'$. Since $\mathcal{K}(f \circ \rho'' \circ \rho')(A')$ is the barycenter of σ, we must have $\beta_{11} + \beta_{12} = \beta_{21} + \beta_{22} = \beta_{31} + \beta_{32}$. Hence by Theorem 7.1, we must have $\beta_{11} + \beta_{12} = \beta_{21} + \beta_{22} = \beta_{31} + \beta_{32} = 1$. Consequently we know, after perhaps exchanging A_1'' and A_2'', that $\mathcal{K}(f \circ \rho'')(A_1'')$ is one of the vertices of σ and $\mathcal{K}(f \circ \rho'')(A_2'')$ is the barycenter of the opposite side. Hence after reordering we may assume that $\mathcal{K}(\rho'')(A_1'') = A_1$ and $\mathcal{K}(f \circ \rho'')(A_2'')$ is the barycenter of B_2B_3.

If $\mathcal{K}(\rho'')(A_2'')$ is a vertex, let this vertex be the sheet A_5. Since $A_1'' \cap A_2''$ is nonempty, we know that $A_1 \cap A_5$ is nonempty also so we are finished. So we are left with the case where $\mathcal{K}(\rho'')(A_2'')$ is not a vertex. Then Lemma 7.3 implies that $A_2 \cap A_3$ is nonempty and $\mathcal{K}(\rho'')(A_2'')$ is the barycenter of the 1-simplex A_2A_3. So at some point in the sequence of blowups ρ'', we blow up the intersection of the strict transforms of A_2 and A_3, and then A_2'' is the strict transform of the resulting exceptional divisor. But then, we would have $A_1 \cap A_2 \cap A_3$ nonempty which is a contradiction. We could also see this geometrically, since $\mathcal{K}(\rho'')$ gives a subdivision of the complex $\mathcal{K}(X,\mathcal{A})$ and this subdivision includes a one simplex from A_1 to the barycenter of A_2A_3. Hence $A_1A_2A_3$ must have been a simplex which is a contradiction.

One can give another proof which the reader may prefer. Pick a 2-simplex δ so that $\mathcal{K}(f)(\delta)$ contains the barycenter of $B_1B_2B_3$. Let δ have vertices $A_6A_7A_8$. Let T_i be the 3×1 vector whose entries are the exponents of A_i in B_1, B_2 and B_3. If T is the 3×3 matrix whose columns are T_6, T_7 and T_8 then we know that T is invertible over the integers so there are unique integers a_i so that $a_6T_6 + a_7T_7 + a_8T_8 = (1,1,1)^T$. Since the barycenter lies in the image of $A_6A_7A_8$ we know that each a_i is nonnegative. Since the entries of T_i are also nonnegative we know that each a_i is either 1 or 0 and if a_i is 0 then all the entries of T_i are either 1 or 0. So if $a_6 = a_7 = a_8 = 1$ then after reordering we must have $A_i = A_{5+i}$ and the first possibility (i) holds. If after reordering we have $a_6 = a_7 = 1$ and $a_8 = 0$ then after more reordering we must have $T_6 = (1,1,0)^T$ and $T_7 = (0,0,1)^T$ and possibility (iii) holds. Finally, if $a_6 = 1$ and $a_7 = a_8 = 0$ then $T_6 = (1,1,1)^T$ and possibility (ii) holds. □

At the moment it is just a hope that the preceding facts will allow you to understand the dimension three case. But they do give powerful information. In particular, let us see what Lemma 7.4 tells us in the three dimensional case. Note that $\mathcal{K}(X, \mathcal{A})$ is two dimensional. Now look at a 2-simplex of $\mathcal{K}(Y, \mathcal{B})$ and apply Lemma 7.4. If the first case holds, we are happy. In the second case, note that the exponent of A_4 in B_i must be 1. To see this, note that these exponents are all equal since we map to the barycenter. But if they have a nontrivial common divisor, it violates the determinant ± 1 condition of Theorem 7.7. It is natural to hope that blowing up $B_1 \cap B_2 \cap B_3$ will improve matters. It may be necessary to blow up X some so that f lifts though, so this is a delicate issue. Good invariants must be found. If the last case holds, we can then blow up Y along the curve $B_2 \cap B_3$ and then f lifts, sort of. Unfortunately, there may be other 2-simplices with the edge $B_2 B_3$ and you must consider them also, but anyway you have hope. I presume that something useful comes out of this anyway, but it is definitely work in progress which I invite others to investigate. I have a computer program which helps compute examples and may be of use, especially on some platforms where one can see a picture of the subdivisions being done. In computer experiments with random examples, I have an algorithm for choosing the center to blow up and this algorithm appears to work well, converging to the desired situation where $\mathcal{K}(f)$ takes vertices to vertices. I have only tested it on random examples with a couple hundred vertices, since in intermediate stages there tend to be several thousand vertices which taxes the limits of my computer, but these results are quite encouraging even though it is not yet clear how to prove the algorithm will always converge.

Finally, since I refer to Proposition 3.3.9 of [**1**] and its proof is somewhat complex, I give a proof which works in the constant exponent case which is needed here. It is based on the alternative proof sketched at the end of the proof in [**1**]. The simplicial complex point of view allows some simplifications to be made.

THEOREM 7.9. *Let* $f\colon (X, \mathcal{A}) \to (Y, \mathcal{B})$ *be an exact tico map with constant exponents and let* $\pi\colon (Y', \mathcal{B}') \to (Y, \mathcal{B})$ *be a tico blowup whose center* C *is an intersection of sheets. Then there is a sequence of tico blowups* $\rho\colon (X', \mathcal{A}') \to (X, \mathcal{A})$ *and a tico map* $f'\colon (X', \mathcal{A}') \to (Y', \mathcal{B}')$ *so that*

(i) $\pi \circ f' = f \circ \rho$.

(ii) *All the centers of the blowups in* ρ *are the intersections of exactly two sheets.*

Proof. Let $B_1, \ldots, B_k$ be the sheets of $\mathcal{B}$ so that $C = \bigcap_{i=1}^{k} B_i$. Let σ be any simplex of $\mathcal{K}(Y, \mathcal{B})$ which contains all the vertices $B_1, \ldots, B_k$. Label its vertices $B_1, \ldots, B_m$. Let $F \subset \sigma$ be the set of points with barycentric coordinates $(t_1 : \ldots : t_m)$ so that $t_1 = t_2$ and $t_1 \leq t_i$ for all $i \leq k$. Then F has codimension one in σ and is the image of a simplex in $\mathcal{K}(Y', \mathcal{B}')$. We will find a sequence of blowups ρ so that for each simplex δ of $\mathcal{K}(X', \mathcal{A}')$, the interior of δ is either disjoint from or contained in $\mathcal{K}(f\rho)^{-1}(F)$. It suffices to do this, since then we may do it for each such simplex σ and each ordering of $B_1, \ldots, B_k$, after which Theorem 7.5 shows that the lifting f' exists.

If A_1A_2 is a 1-simplex of $\mathcal{K}(X, \mathcal{A})$ define a nonnegative integer $\gamma(A_1A_2)$ as follows. Let $F' \supset F$ be the set of points in σ with barycentric coordinates $(t_1 : \ldots : t_m)$ so that $t_1 = t_2$. Let α_{ij} be the exponent of A_i in B_j. We set $\gamma(A_1A_2) = 0$ if the interior of A_1A_2 does not intersect $\mathcal{K}(f)^{-1}(F')$ or if $\mathcal{K}(f)(A_1A_2) \subset F$. Otherwise, after perhaps reordering we must have $\alpha_{11} > \alpha_{12}$ and $\alpha_{21} < \alpha_{22}$. We then let $\gamma(A_1A_2) = \alpha_{11} - \alpha_{12} + \alpha_{22} - \alpha_{21}$. If δ is any simplex of $\mathcal{K}(X, \mathcal{A})$, we let $\gamma(\delta)$ be the maximum of $\gamma(\delta_1)$ over all 1-simplices δ_1 in δ and we let $\lambda(\delta)$ be the number of 1-simplices δ_1 in δ with $\gamma(\delta_1) = \gamma(\delta)$.

The proof now proceeds as follows. Among all simplices δ of $\mathcal{K}(X, \mathcal{A})$, pick one with lexicographically maximum $(\gamma(\delta), \lambda(\delta))$. If this maximum has $\gamma(\delta) = 0$ then we are done. If it is not 0, we may pick vertices A_1 and A_2 of δ so that $\gamma(A_1A_2)$ is this maximum γ. We then blow up $A_1 \cap A_2$. After blowing up, either the maximum (γ, λ) has decreased, or there are fewer simplices δ with the maximum (γ, λ). Consequently, after a finite number of such steps we are done. □

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