# Poincaré-Hopf Theorems on Singular Spaces

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ABSTRACT. In 1991 M.-H. Schwartz proved a Poincaré-Hopf theorem for stratified radial vector fields on Whitney stratified analytic varieties with boundary. In this paper we extend her theory and results to a wide class of stratified sets with boundary, called mapping cylindered manifold complexes, which include all Thom-Mather stratified sets, but also arbitrary (locally) finite partitions of subanalytic sets into subanalytic submanifolds. In the process we give a short proof of a generalisation of the Poincaré-Hopf theorem for radial vector fields of M.-H. Schwartz (our Theorem 2).

We then study more general stratified vector fields which need not be radial and may further be tangent to the boundary of the manifold complex, and prove the corresponding Poincaré-Hopf theorems. In particular, a notion is given of the virtual index of a vector field on a mapping cylindered manifold complex and the Euler characteristic of the complex is shown to be the sum of the virtual indices.

**Introduction.**

The classical Poincaré-Hopf theorem for a vector field $v$ with only isolated zeros on a smooth compact manifold $M$ without boundary says that the Euler characteristic of $M$ equals the sum of the indices of $v$ at its zeros. M.-H. Schwartz has developed a theory ([**S1**], [**S2**], [**S3**]) giving an analogous formula for the Euler characteristic of a Whitney stratified (singular) analytic variety $V$ of an analytic manifold $M$ as the sum of the indices of zeros on $V$ of a "radial" vector field $v$ on $M$ which is tangent to the strata of $V$. The index of a zero on a stratum $X$ is defined to be the index of the restriction of $v$ to $X$. The "radial" vector fields of M. H. Schwartz are defined in terms of two types of families of tubes around strata, one type called parametric tubes, given by barycentric subdivision of a triangulation, and the other called geodesic tubes, given by certain geodesic tubular neighbourhoods defined using the ambient metric. Such a vector field is exiting from sufficiently small geodesic tubes around a stratum $X$ over some closed subset of $X$ which contains the zeros of $v$ on $X$. Having two types of families of tubes without compatibility relations makes proving the existence of M. H. Schwartz's "radial" fields quite complicated, and further requires delicate extension arguments and angle estimates when adapting the "radial" vector fields to yield "exiting" vector fields in a suitable form for the Poincaré-Hopf theorem (see [**S3**]).

Now it is well-known that Whitney stratified sets (not necessarily analytic) are known to possess compatible systems of tubes around strata, and thus they are Thom-Mather stratified sets, and also that all Thom-Mather stratified sets have a mapping cylinder stratified structure [**T**], [**Ma**]. We exploit this to define a simple notion of radial stratified vector field on abstract stratified sets which have such a mapping cylinder stratified structure. In fact the theory of radial vector fields works well for a class of "mapping cylinder manifold complexes" (defined below) which are not necessarily even stratified sets in the

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classical sense, although we will continue to often refer to the "pieces" of the manifold complex as the "strata" and the complex as a whole as a stratified set. Our mapping cylinder manifold complexes are thus much more general than Whitney stratified sets or Thom-Mather abstract stratified sets. For example, any locally finite subanalytic filtration of a subanalytic set is a mapping cylinder manifold complex in our sense. In particular our mapping cylinder manifold complexes are not required to have strata of increasing dimensions in successive skeleta, nor are they necessarily locally topologically trivial along strata. In general, our mapping cylinder manifold complexes are not even triangulable, whereas Thom-Mather stratified sets are triangulable [**G**], as are arbitrary locally finite subanalytic partitions [**H**].

The proofs of existence of our radial vector fields (Theorem 1), and of the Poincaré-Hopf formula (Theorem 2), become straightforward and short. We use induction on the depth of strata, rather than on their dimension. This produces much shorter proofs. Now observe that in the abstract context of our theory, ambient continuity of stratified vector fields has no meaning. As a consequence, when our results are applied to embedded stratified sets in a manifold, they hold for stratified vector fields which are not continuous, unlike those of M. H. Schwartz whose "radial" vector fields are always required to be continuous on the ambient manifold. Natural examples of non-continuous stratified vector fields are the projection onto strata of a given vector field on $M$, in particular the gradient vector field on strata which is defined by a stratified Morse function [**GM1**].

In the present paper we show that the Poincaré-Hopf formula applies moreover to what we call semiradial vector fields (Corollary 6) – roughly those which are not vertically descending near strata with respect to the mapping cylinder structure. Further generalisation is obtained by defining a virtual index at points on a stratum $X$ which are limits of points where $v$ is vertically descending towards $X$ ; we call such limit points virtual zeros. The Euler characteristic of the manifold complex is shown to equal the sum of the indices and virtual indices of a stratified vector field whose zeros and virtual zeros are isolated (Theorem 10). This applies to generic vector fields in many embedded situations.

Another improvement we obtain concerns behaviour of $v$ on the boundary of the manifold complex, defined to be the union of the boundaries of strata (assumed to be manifolds with boundary). We allow tangency, and deduce as a special case the Morse-Pugh theorem (see the discussion after Theorem 3) which gives a Poincaré-Hopf formula for generic vector fields on a manifold with boundary [**Mo**], [**P**]. We note that V.Arnol'd [**A**], again for manifolds with boundary, gave a refined Poincaré-Hopf index formula for zeros of 1-forms. It would be interesting to understand the relation between the index of Arnol'd and our virtual index. We thank M. Shub and J. Dupont for respectively indicating to us the papers of Pugh and Morse.

To sum up, we extend the theory of M.-H. Schwartz in four ways :

1) Our stratified sets (mapping cylindered manifold complexes) need not be analytic, nor Whitney regular, nor even possess a Thom-Mather structure. They include arbitrary locally finite subanalytic decompositions of subanalytic sets into smooth manifolds.
2) Our vector fields are not required to be continuous in the ambient space, even when this makes sense. Continuity was essential for the proof given by M.-H. Schwartz, where the stratified vector field is extended to a continuous vector field on a neighbourhood

of the stratified set so as to be able to apply the classical Poincaré-Hopf theorem.

3) The vector fields we consider are not necessarily radial : we first allow semi-radiality, and then remove this condition, requiring only that both virtual zeros and zeros are isolated.
4) We do not require that our vector fields be transverse to the boundary (i.e. be either exiting or entering) : tangency is allowed.

The classical Poincaré-Hopf theorem for vector fields on smooth manifolds (with boundary) can be deduced from the Lefschetz fixed-point theorem. Goresky and MacPherson (in **[GM2]**) have recently given generalised Lefschetz fixed point formulae for self-maps of stratified subanalytic sets. Our theory applies to more general sets, but even when the objects coincide, Lefschetz fixed-point formulae usually cannot be used to give Poincaré-Hopf index formulae because most stratified vector fields are not globally integrable, so will not generate self-maps.

## Mapping cylinder manifold complexes

The objects we will be considering are stratified sets of a certain general type. For the purposes of this paper, a **manifold complex with boundary** is a locally compact Hausdorff topological space $X$ and a finite nested collection of closed subsets $X_0 \subset X_1 \subset \cdots \subset X_m = X$ so that if $S_i = X_i - X_{i-1}$ then:

1) each $S_i$ has the structure of a smooth manifold with boundary $\partial S_i$,
2) $\mathrm{Cl}(\partial S_i) \subset \bigcup_{j=0}^{i} \partial S_j$.

We call $X_i$ the **skeleta** of $X$ and $S_i$ the **strata** of $X$. We let $\partial X$ denote $\bigcup_{j=0}^{m} \partial S_j$. Note that $\partial X$ has the natural structure of a manifold complex with empty boundary. We say the **depth** of $X$ is $m' - \ell$ where $m'$ is the smallest index so that $X_{m'} = X$ and $\ell$ is the smallest index so that $X_\ell$ is not empty. Note that the strata $S_i$ need not have dimension $i$. In fact the dimensions of $S_i$ need not be monotonically increasing. For example, any finite CW complex is a manifold complex (with empty boundary).

An open subset of a manifold complex inherits the structure of a manifold complex in the natural way (skeleta and strata are just intersected with the open set). Likewise there is a natural manifold complex structure on the product of a manifold complex with a manifold.

The above notion of manifold complex is a bit too general to work with conveniently. The strata might approach each other in a very complicated way. So we add another property. In the manifold complexes we will consider, strata will have mapping cylinder neighborhoods. For example Thom-Mather stratified sets have the properties alluded to below. On a first reading the reader may prefer to think only of the more familiar Thom-Mather stratified sets or Whitney stratified sets rather than the more general manifold complexes defined below.

We say that $X$ is a **mapping cylindered manifold complex with boundary** if it is a manifold complex with boundary and the strata $S_i$ have compatible mapping cylinder neighborhoods. That is, for each stratum $S_i$ there is an open neighborhood $U_i$ of $S_i$ in $X$, a manifold complex with boundary $L_i$, a neighborhood $V_i$ of $L_i \times 0$ in $L_i \times [0,1)$ and a proper continuous map $q_i\colon V_i \to U_i$ so that:

1) $q_i^{-1}(S_i) = L_i \times 0$ and the restriction of $q_i$ to $V_i - L_i \times 0$ is an isomorphism onto $U_i - X_i$, i.e., it is a homeomorphism which maps each stratum of $V_i - L_i \times 0$ to a stratum of $U_i - X_i$.
2) $q_i^{-1}(\partial X) = \partial L_i \times [0,1) \cap V_i$.
3) For all $i < j$ and $(x,t) \in q_j^{-1}(U_i)$ and $(x,s) \in q_j^{-1}(U_i)$ we have $\rho_i q_j(x,t) = \rho_i q_j(x,s)$ where $\rho_i$ is the mapping cylinder level function defined below.
4) The restriction of $q_i$ to each stratum of $V_i - L_i \times 0$ is a diffeomorphism onto a stratum of $U_i - X_i$

There are some useful functions which come from this mapping cylinder structure on $U_i$. We have a retraction $\pi_i: U_i \to S_i$ defined by $\pi_i q_i(x,t) = q_i(x,0)$ and $\pi_i(p) = p$ for all $p \in S_i$. We also have a level function $\rho_i: U_i \to [0,1)$ defined by setting $\rho_i q_i(x,t) = t$ making $\rho_i(p) = 0$ for $p \in S_i$. Note that $\pi_i^{-1}(\partial S_i) = U_i \cap \partial X$. Also $\rho_i \pi_j = \rho_i$ if $j > i$. The restriction of $\rho_i$ to each stratum is smooth, although $\pi_i$ need not be smooth on a stratum. Properness of $q_i$ implies that $\pi_i$ and $\rho_i$ are continuous.

For example, a Whitney stratified set is a mapping cylindered manifold complex with boundary. The maps $\pi_i$ and $\rho_i$ are just the usual tubular data and $q_i$ is obtained by integrating a controlled vector field $\xi_i$ on $U_i - S_i$ with $d\rho_i(\xi_i) = 1$ and $d\pi_i(\xi_i) = 0$.

Note that $\partial X$ inherits the structure of a mapping cylindered manifold complex with boundary by just restricting each $q_i$ to $\partial L_i \times [0,1) \cap V_i$. We do not expect $\partial X$ to have a collar neighborhood in general.

For various technical reasons, (mostly having to do with corner rounding), it is useful to generalize the notion of mapping cylindered manifold complex with boundary a bit. In particular, a **weak mapping cylindered manifold complex with boundary** will satisfy conditions 1) through 3) above, but instead of 4), will satisfy

5) For each $j > i$, the restriction of $\rho_i$ to $U_i \cap S_j$ is smooth, with no critical points. Moreover, there are vector fields $\xi_i$ on $U_i - S_i$ tangent to $\partial X$ so that $d\rho_i(\xi_i) = 1$ and $d\rho_k(\xi_i) = 0$ for all $k < i$.

To see that a mapping cylindered manifold complex with boundary satisfies 5), let $\xi_i'$ be the vector field $0 \times 1$ on $L_i \times (0,1)$ and let $\xi_i$ be the vector field on $U_i - X_i$ which is $dq_i(\xi_i')$. Then 3) implies that $d\rho_k(\xi_i) = 0$ for all $k < i$.

It is straightforward to prove that an arbitrary locally finite subanalytic partition of a subanalytic set into smooth submanifolds has a weak mapping cylindered manifold complex with boundary structure. (For a proof one can use a subanalytic triangulation, or a subanalytic Whitney stratification compatible with the given structure.)

Apparently, stratified sets with boundary were first considered in [**AK2**]; they were also used in [**V**]. Siebenmann was the first to use the useful notion of depth in [**Sie**].

## Poincaré-Hopf theorems for radial vector fields

From now on $X$ will be a weak mapping cylindered manifold complex with boundary with the various data $U_i$, $V_i$, $q_i$, $\xi_i$, $\rho_i$ and $\pi_i$ as above. For convenience, extend each $\xi_i$ (noncontinuously) to be zero outside $U_i - X_i$.

Let $\mathbf{v}$ be a **vector field** on $X$, i.e., for each $x \in X$ if $x \in S_i$ then $\mathbf{v}(x)$ is a vector in the tangent space to $S_i$ at $x$. Furthermore, we ask that the restriction of $\mathbf{v}$ to each stratum $S_i$ be continuous.

Suppose that the zeroes of $\mathbf{v}$ are isolated. Then to each point $p \in S_i - \partial S_i$ we may assign an index $\mathcal{I}(p, \mathbf{v})$ by restricting $\mathbf{v}$ to $S_i$ and taking the usual index (c.f., [**Mi**]). Note that $\mathcal{I}(p, \mathbf{v}) = 0$ unless $\mathbf{v}(p) = 0$. For convenience we also set $\mathcal{I}(p, \mathbf{v}) = 0$ for $p \in \partial X$ where $\mathbf{v}(p) \neq 0$ and leave $\mathcal{I}(p, \mathbf{v})$ undefined for $p$ a zero of $\mathbf{v}$ on $\partial X$.

We say that a vector field $\mathbf{v}$ is **radial** if for each $i$ there is a neighborhood $U_i'$ of $S_i$ in $U_i$ so that $d\rho_i(\mathbf{v}(x)) > 0$ for all $x \in U_i' - S_i$.

**Lemma 1** *Any weak mapping cylindered manifold complex with boundary $X$ has a radial vector field $\mathbf{v}$ with isolated zeroes which points inward on $\partial X$.*

**Proof:** Pick any vector field $\mathbf{v}'$ on $X$ which points inward on $\partial X$. We suppose by induction on $k$ that we have constructed nonnegative functions $\beta_i \colon X \to [0, \infty)$ for $0 \leq i < k$ so that:

a) $\beta_i$ is continuous on each stratum.

b) $\beta_i$ vanishes on $X_i$ and $X - U_i$ and in fact, the support of the restriction of $\beta_i$ to any stratum $S_j$ is contained in $U_i \cap S_j$. Thus $\beta_i \xi_i$ is continuous on each stratum of $X$.

c) For each $j < k$, there is a neighborhood $U_j'$ of $S_j$ so that $d\rho_j(\mathbf{v}' + \sum_{i=0}^{k-1} \beta_i \, \xi_i) > 0$ on $U_j' - S_j$.

It is easy to choose a suitable $\beta_k$ to complete the induction. We only need choose $\beta_k$ continuous on each stratum, vanishing on $X_k$, with support in $U_k \cup X_k$, thus giving a) and b), so that for some neighborhood $U_k'$ of $S_k$ we have

$$\beta_k(x) > -d\rho_k(\mathbf{v}'(x)) - \sum_{i=0}^{k-1} \beta_i(x) d\rho_k(\xi_i(x))$$

for all $x \in U_k' - S_k$. Since $d\rho_j(\beta_k \xi_k) = 0$ for all $j < k$, we know c) still holds.

So we may assume $k = m$. Let $\mathbf{v}''$ be the vector field $\mathbf{v}' + \sum_{i=0}^{m-1} \beta_i \xi_i$. Then $\mathbf{v}''$ is radial because of c) and it points inward on $\partial X$ because $\mathbf{v}'$ did and each $\xi_i$ is tangent to $\partial X$. We could almost let $\mathbf{v}$ be $\mathbf{v}''$, but $\mathbf{v}''$ might not have isolated zeroes. However, c) implies that the zeroes of $\mathbf{v}''$ in any stratum $S_i$ must lie in $S_i - \bigcup_{j<i} U_j'$. So we may, after perhaps slightly shrinking the neighborhoods $U_j'$, let $\mathbf{v}$ restricted to any stratum $S_i$ be any vector field which points inward on $\partial S_i$, has isolated zeroes and equals $\mathbf{v}''$ on $\bigcup_{j<i} U_j' \cap S_i$.

**Q.E.D.**

A similar result for Thom-Mather stratified sets was proved independently by Simon ([**Sim**]).

We now give our first Poincaré-Hopf theorem. As an example of its application, let $X$ be a finite simplicial complex, the strata being the interiors of simplices. Take the vector field $\mathbf{v}$ which on each simplex points directly to the barycenter of that simplex. Then each simplex $\sigma$ contains exactly one zero of $\mathbf{v}$ (at the barycenter) and its index is $(-1)^{\dim \sigma}$. Thus Theorem 2 gives the usual formula for the Euler characteristic of $X$.

**Theorem 2** *Let $X$ be a compact weak mapping cylindered manifold complex with boundary. Then the singular homology groups $H_*(X)$ and $H_*(\partial X)$ are finitely generated, so the Euler characteristics $\chi(X)$ and $\chi(\partial X)$ are defined. Suppose $\mathbf{v}$ is a radial vector field on $X$, the zeroes of $\mathbf{v}$ are isolated and they do not occur on the boundary of $X$.*

1) *If* $\mathbf{v}$ *points inward on* $\partial X$ *then* $\chi(X) - \chi(\partial X) = \sum_{p\in X} \mathcal{I}(p,\mathbf{v})$.
2) *If* $\mathbf{v}$ *points outward on* $\partial X$ *then* $\chi(X) = \sum_{p\in X} \mathcal{I}(p,\mathbf{v})$.
3) *In fact, if* $\mathbf{v}$ *points inward on some components* $\partial X_-$ *of* $\partial X$ *and outward on the others, then* $\chi(X) - \chi(\partial X_-) = \sum_{p\in X} \mathcal{I}(p,\mathbf{v})$.

**Proof:** We will first prove finite generation and part 1), and then the other results will follow easily. We prove finite generation and 1) by induction on the depth of $X$.

Let $k$ be the smallest index so that $X_k$ is not empty. Then $X_k = S_k$ is compact so there is an $\epsilon > 0$ so that $U'_k \supset \rho_k^{-1}([0,\epsilon])$. (Recall that $U'_k$ is such that $d\rho_k(\mathbf{v}) > 0$ on $U'_k - S_k$ so we know that $\mathbf{v}$ has no zeroes on $\rho_k^{-1}((0,\epsilon])$.) Let $X' = X - \rho_k^{-1}([0,\epsilon))$ and let $Y = \rho_k^{-1}([0,\epsilon])$.

**Claim 2.1** *$X'$ can be given the structure of a weak mapping cylindered manifold complex with boundary (by rounding corners). Furthermore,* $\mathbf{v}$ *points inward on $\partial X'$ and the depth of $X'$ is less than that of $X$. Also $\partial X' = (\partial X \cap X') \cup \rho_k^{-1}(\epsilon)$.*

The proof of claim 2.1 is just corner rounding. Set $X'_i = X_i \cap X'$ and let $q'_i$, $\xi'_i$, $\pi'_i$ and $\rho'_i$ be the restrictions of $q_i$, $\xi_i$, $\pi_i$ and $\rho_i$ for $i > k$ and empty for $i \le k$. Then the stratum $S'_i = S_i \cap X'$ is a smooth manifold with boundary $(\partial S_i \cap X') \cup (S_i \cap \rho_k^{-1}(\epsilon))$ and a corner on the boundary at $\partial S_i \cap \rho_k^{-1}(\epsilon)$. Just smooth out this corner. The restriction of $\mathbf{v}$ to $X'$ still points inward on $\partial X'$ since $\mathbf{v}$ points inward on $\partial X$ and $d\rho_k(\mathbf{v}) > 0$ on $\rho_k^{-1}(\epsilon)$. The vector fields $\xi'_i$ are tangent to $\partial X'$ since $d\rho_k(\xi_i) = 0$. Note that 5) in the definition of weak mapping cylindered manifold complex with boundary implies that for any collection of indices $i_1, \ldots, i_\ell$ the map $(\rho_{i_1}, \ldots, \rho_{i_\ell})\colon U_{i_1} \cap \ldots \cap U_{i_\ell} \to \mathbf{R}^\ell$ is a submersion, and restricts to a submersion on the corner of $X'$. Consequently, we can round corners and still keep the $\rho_i$ smooth, after perhaps shrinking the $U_i$.

**Claim 2.2** *$Y$ deformation retracts to $X_k$.*

**Claim 2.3** *$Y \cap \partial X$ deformation retracts to $\partial X_k$.*

The deformation retraction for claim 2.2 is $H\colon Y \times [0,1] \to Y$ given by $H(x,t) = x$ for $x \in X_k$ and $H(q_k(y,s),t) = q_k(y,st)$ for $y \in L_k$ and $s \in [0,\epsilon]$. Its restriction gives the deformation retraction of claim 2.3.

Now by claim 2.1 and induction on depth we have $H_*(X')$ and $H_*(\partial X')$ finitely generated and:

$$\chi(X') - \chi(\partial X') = \sum_{p\in X'} \mathcal{I}(p,\mathbf{v}) = \sum_{p\in X - X_k} \mathcal{I}(p,\mathbf{v})$$

since $\mathcal{I}(p,\mathbf{v}) = 0$ for all $p \in \rho_k^{-1}((0,\epsilon])$ because $\mathbf{v}(p) \neq 0$. Also by claim 2.2 and claim 2.3 and the classical Poincaré-Hopf theorem for manifolds we have

$$\sum_{p\in X_k} \mathcal{I}(p,\mathbf{v}) = \chi(X_k) - \chi(\partial X_k) = \chi(Y) - \chi(Y \cap \partial X).$$

Thus we have

$$\sum_{p\in X} \mathcal{I}(p,\mathbf{v}) = \chi(X') - \chi(\partial X') + \chi(Y) - \chi(Y \cap \partial X).$$

Note $H_*(Y)$ is finitely generated since it deformation retracts to the compact manifold $X_k$. Also $X' \cap Y = \rho_k^{-1}(\epsilon)$ has finitely generated homology, since it is the deformation retract of $\rho_k^{-1}([\epsilon/2, \epsilon])$ which after corner rounding is a weak mapping cylindered manifold complex with boundary of smaller depth than $X$. The map $q_k$ gives a bicollaring of $X' \cap Y$, so the Mayer-Vietoris sequence for $X'$ and $Y$ is exact, which implies that $H_*(X)$ has finitely generated homology and

$$\chi(X) = \chi(X') + \chi(Y) - \chi(\rho_k^{-1}(\epsilon)).$$

Thus

$$\sum_{p \in X} \mathcal{I}(p, \mathbf{v}) = \chi(X) - \chi(\partial X') + \chi(\rho_k^{-1}(\epsilon)) - \chi(Y \cap \partial X).$$

But $Y \cap \partial X \cap X'$ is bicollared in $\partial X$ so

$$\chi(\partial X) = \chi(\partial X \cap X') + \chi(Y \cap \partial X) - \chi(Y \cap \partial X \cap X').$$

If $Z = \rho_k^{-1}(\epsilon) \cup (\partial X \cap \rho_k^{-1}([\epsilon, \epsilon')))$ for $\epsilon'$ slightly bigger than $\epsilon$, then $Z$ deformation retracts to $\rho_k^{-1}(\epsilon)$ and $Z \cap X' \cap \partial X$ deformation retracts to $\rho_k^{-1}(\epsilon) \cap \partial X$. But the Mayer-Vietoris sequence for $Z$ and $X' \cap \partial X$ is exact, so

$$\chi(\partial X') = \chi(\partial X \cap X') + \chi(\rho_k^{-1}(\epsilon)) - \chi(\partial X \cap \rho_k^{-1}(\epsilon)).$$

However

$$Y \cap \partial X \cap X' = \partial X \cap \rho_k^{-1}(\epsilon)$$

so we get

$$\chi(\partial X) = \chi(\partial X') + \chi(Y \cap \partial X) - \chi(\rho_k^{-1}(\epsilon)).$$

So

$$\sum_{p \in X} \mathcal{I}(p, \mathbf{v}) = \chi(X) - \chi(\partial X)$$

and we have proven part 1).

Now let us prove part 3), given part 1). Note that 3) easily implies 2). Using Lemma 1, pick a radial vector field $\mathbf{v}'$ on $X$ with isolated zeroes which points inward on all of $\partial X$. Let $\partial X_+ = \partial X - \partial X_-$ and let $X''$ be the weak mapping cylindered manifold complex with boundary $X'' = X \times 0 \cup \partial X_+ \times [0, 1] \cup X \times 1$. The strata of $X''$ are $S_i'' = S_i \times 0 \cup (\partial X_+ \cap \partial S_i) \times [0, 1] \cup S_i \times 1$.

We have a vector field $\mathbf{v}''$ on $X''$ which we take to be $\mathbf{v}$ on $X \times 0$ and $\mathbf{v}'$ on $X \times 1$ and interpolate linearly between $\mathbf{v}$ and $\mathbf{v}'$ on $\partial X_+ \times [0, 1]$. Since $\mathbf{v}$ points outward on $\partial X_+$ and $\mathbf{v}'$ points inward on $\partial X_+$, the vector field $\mathbf{v}''$ has no zeroes on $\partial X_+ \times [0, 1]$. Note that $\mathbf{v}''$ points inward on all of $\partial X'' = \partial X_- \times \{0, 1\}$. So by part 1) we have:

$$\begin{aligned}
\chi(X'') - \chi(\partial X'') &= \sum_{p \in X''} \mathcal{I}(p, \mathbf{v}'') \\
&= \sum_{p \in X} \mathcal{I}(p, \mathbf{v}) + \sum_{p \in X} \mathcal{I}(p, \mathbf{v}') \\
&= \sum_{p \in X} \mathcal{I}(p, \mathbf{v}) + \chi(X) - \chi(\partial X).
\end{aligned}$$

But $\partial X''$ is just two copies of $\partial X_-$ so we get

$$\sum_{p\in X} \mathcal{I}(p,\mathbf{v}) = \chi(X'') - 2\chi(\partial X_-) - \chi(X) + \chi(\partial X)$$

$$= \chi(X'') - \chi(\partial X_-) - \chi(X) + \chi(\partial X_+).$$

However by Mayer-Vietoris we get

$$\chi(X'') = \chi(X) + \chi(X) - \chi(\partial X_+)$$

and so

$$\sum_{p\in X} \mathcal{I}(p,\mathbf{v}) = \chi(X) - \chi(\partial X_-).$$

**Q.E.D.**

The above Theorem 2 implies Theorem 6.2.2 of [**S3**]. This theorem says that if $W$ is a Whitney stratified analytic subset of an analytic manifold $M$ and $D \subset M$ is a compact codimension 0 submanifold of $M$ whose boundary is transverse to the strata of $W$ and if $X^{+rad}$ (resp. $X^{-rad}$) is a radial vector field on $D$ (in the sense of [**S3**]) which is tangent to the strata of $W$ and points outward (resp. inward) on $\partial D$ then we have

$$\chi(W \cap D) = \sum_{p\in W\cap D} \mathcal{I}(p, X^{+rad})$$

$$\chi(W \cap D) - \chi(W \cap \partial D) = \sum_{p\in W\cap D} \mathcal{I}(p, X^{-rad}).$$

If we choose tubular data for $W$ so that $\rho_i$ is the distance from $S_i$ (which is in fact what one normally does) then the notion of radial in [**S3**] is the same as the one we give here. So Theorem 6.2.2 in [**S3**] follows from Theorem 2 above since $W \cap D$ is a mapping cylindered manifold complex with boundary $W \cap \partial D$.

Note however the greater generality of Theorem 2. It applies to other than analytic sets, also the vector fields it applies to need not have a continuous extension to an ambient manifold $M$.

We also have a theorem in the more general case where we do not restrict the behavior of $\mathbf{v}$ on $\partial X$ except to ask that it have no zeroes. But first we need some definitions.

We say that a vector field $\mathbf{v}'$ on $\partial X$ is a **radial projection** of $\mathbf{v}$ to $\partial X$ if it is radial (on $\partial X$), has isolated zeroes and also for each $x \in \partial X$ so that $\mathbf{v}(x)$ is tangent to $\partial X$ we know that $\mathbf{v}'(x) = \mathbf{v}(x)$.

We will show in Lemma 4 below that any radial vector field has a radial projection to $\partial X$. Meanwhile we get the following theorem which easily implies Theorem 2.

**Theorem 3** *Suppose $\mathbf{v}$ is a radial vector field on a compact weak mapping cylindered manifold complex with boundary $X$, the zeroes of $\mathbf{v}$ are isolated and do not occur on the boundary of $X$. Let $\mathbf{v}'$ be a radial projection of $\mathbf{v}$ to $\partial X$. Then*

$$\chi(X) = \sum_{p\in X} \mathcal{I}(p,\mathbf{v}) + \sum_{q\in A} \mathcal{I}(q,\mathbf{v}')$$

*where $A$ is the set of points $x \in \partial X$ such that $\mathbf{v}(x)$ points into $X$.*

**Proof:** We prove this by modifying $X$ and $\mathbf{v}$ until we reach the point where we may apply Theorem 2. Let $X' = X \cup (\partial X \times [-1,1])$ where we identify each point $x \in \partial X$ with $(x,-1) \in \partial X \times [-1,1]$. We take the natural stratification of $X'$ by gluing the collar $\partial S_i \times [-1,1]$ onto each stratum $S_i$ and picking a smooth structure on the result (so that $S_i$ and $\partial S_i \times [-1,1]$ are smooth submanifolds). We give $X'$ the structure of a weak mapping cylindered manifold complex with boundary in the natural way. In particular its radial and projection functions when restricted to $X$ are the same as for $X$. On $\partial X \times [-1,1]$ they are the obvious extensions. As usual, the submersive property of the $\rho_i$ allow us to choose a smoothing so that the new $\rho_i$ are still smooth.

We now extend $\mathbf{v}$ to a radial vector field $\mathbf{v}''$ on $X'$ as follows. Note that $\mathbf{v}$ is defined on $\partial X \times -1$, let it be $\mathbf{v}(x) = (\mathbf{v}'''(x), \alpha(x))$ where $\mathbf{v}'''$ is a vector field on $\partial X$ and $\alpha$ is a function $\alpha\colon \partial X \to \mathbf{R}$. Note that $\alpha(x) < 0$ where $\mathbf{v}$ points into $X$, $\alpha(x) > 0$ where $\mathbf{v}$ points out of $X$ and $\alpha(x) = 0$ where $\mathbf{v}$ is tangent to $\partial X$.

On $\partial X \times [-1,1]$ we let

$$\mathbf{v}''(x,t) = \begin{cases} \big((1+t)\mathbf{v}'(x) - t\mathbf{v}'''(x),\ \alpha(x)\big) & \text{if } t \le 0 \\ \big(\mathbf{v}'(x),\ (1-t)\alpha(x) + t\big) & \text{if } t \ge 0. \end{cases}$$

Note that $\mathbf{v}''(x,1) = (\mathbf{v}'(x),1)$ so $\mathbf{v}$ points outward on $\partial X' = X \times 1$. Let us now find the zeroes of $\mathbf{v}''$. On $\partial X \times [-1,0]$ there are no zeroes of $\mathbf{v}''$ since we would have to have $\alpha(x) = 0$ (so $\mathbf{v}(x)$ is tangent to $\partial X$) but then $\mathbf{v}''(x) = (\mathbf{v}(x), 0) \neq 0$. On $\partial X \times [0,1]$, the zeroes occur exactly when $\mathbf{v}'(x) = 0$ and $\alpha(x) = -t/(1-t)$ (or equivalently $t = \alpha(x)/(\alpha(x)-1)$) which means that $\alpha(x) < 0$ so $\mathbf{v}(x)$ points inward. In other words there is one zero $(x_0, t_0)$ on $\partial X \times [-1,1]$ for each point $x_0 \in A$ where $\mathbf{v}'(x_0) = 0$.

**Claim 3.1** *The index of this zero $(x_0,t_0)$ of $\mathbf{v}''$ is exactly the index of $\mathbf{v}'$ at $x_0$.*

To prove claim 3.1, let $N_0$ be a neighborhood of $x_0$ in $\partial X$ containing no other zeroes of $\mathbf{v}'$. Let $\tilde{\mathbf{v}}'$ be a perturbation of $\mathbf{v}'$ in $N_0$ such that $\tilde{\mathbf{v}}'$ has only nondegenerate zeroes in $N_0$. Then the total index of $\tilde{\mathbf{v}}'$ in $N_0$ equals the index of $\mathbf{v}'$ at $x_0$. The associated vector field $\tilde{\mathbf{v}}''(x,t) = (\tilde{\mathbf{v}}'(x),\ (1-t)\alpha(x)+t)$ also has only nondegenerate zeroes in $N_0 \times [0,1]$ and the total index of $\tilde{\mathbf{v}}''$ in $N_0 \times [0,1]$ equals the index of $\mathbf{v}''$ at $(x_0,t_0)$. Now by Lemma 6.4 of **[Mi]** the index of each zero $(x_1,t_1)$ of $\tilde{\mathbf{v}}''$ equals the index of $\tilde{\mathbf{v}}'$ at $x_1$ and the claim follows.

Note that $X'$ deformation retracts to $X$ so by claim 3.1 and Theorem 2,

$$\chi(X) = \chi(X') = \sum_{p \in X'} \mathcal{I}(p, \mathbf{v}'') = \sum_{p \in X} \mathcal{I}(p, \mathbf{v}) + \sum_{q \in A} \mathcal{I}(q, \mathbf{v}').$$

Consequently the theorem is proven. **Q.E.D.**

The above Poincaré-Hopf formula in Theorem 3 appears to be new even for the case where $X$ is a manifold, although **[P]** gave an alternative formula, rediscovered by Pugh 40 years after it first appeared in a paper of Morse (**[Mo]**)! In **[P]**, it is assumed that $X$ is a manifold with boundary and $\mathbf{v}$ is generic so if $\Gamma_1$ is the set of $x \in \partial X$ where $\mathbf{v}$ is tangent to $\partial X$, then $\Gamma_1$ is a smooth codimension one submanifold of $\partial X$, and if $\Gamma_2$ is the set of

$x \in \Gamma_1$ where $\mathbf{v}$ is tangent to $\Gamma_1$ then $\Gamma_2$ is a smooth codimension one submanifold of $\Gamma_1$ and so on. Pugh then gives a Poincaré-Hopf formula with correction terms involving these $\Gamma_i$'s. In particular,

$$\sum_{p \in X} \mathcal{I}(p, \mathbf{v}) = \chi(X) - \chi(\partial X) + \sum \chi(A_i) - \chi(\Gamma_i)$$

where $A_1$ is the set of points in $\partial X$ at which $\mathbf{v}$ is pointing out of $\partial X$ and $A_{i+1}$ is the set of points of $\Gamma_i$ where $\mathbf{v}$ is pointing out of $A_i$. Note $\partial A_i = \Gamma_i$. By Theorem 3 we have $\sum_{p \in X} \mathcal{I}(p, \mathbf{v}) = \chi(X) - \chi(\partial X) + \sum_{q \in A_1} \mathcal{I}(q, \mathbf{v}')$, but by induction we have $\sum_{q \in A_1} \mathcal{I}(q, \mathbf{v}') = \chi(A_1) - \chi(\Gamma_1) + \sum_{i \geq 2} \chi(A_i) - \chi(\Gamma_i)$, so we obtain the formula in **[P]** and **[Mo]**.

## The Poincaré-Hopf theorem for semiradial vector fields

The next thing to do is to try to eliminate the radial condition. We do so by introducing virtual zeroes of $\mathbf{v}$ which are places where the behavior of $\mathbf{v}$ in a neighborhood of a stratum requires some contribution to the index.

We let $\mathbf{B}_i(\mathbf{v})$ be the set of $x \in U_i - X_i$ so that for some numbers $c_j \leq 0$ for $i \leq j \leq m$ we have

$$\mathbf{v}(x) = \sum_{j=i}^{m} c_j \xi_j(x)$$

and $c_i < 0$. Then we say $x \in S_i$ is a **virtual zero** of $\mathbf{v}$ if $x \in \mathrm{Cl}(\mathbf{B}_i(\mathbf{v}))$.

A radial vector field has no virtual zeroes because near $X_i$ we must have $0 < d\rho_i(\mathbf{v}) = c_i$ if $\mathbf{v}(x) = \sum_{j=i}^{m} c_j \xi_j(x)$. Also, if $X$ is a Thom stratified set and $\mathbf{v}$ is a controlled vector field then it has no virtual zeroes since $0 = d\rho_i(\mathbf{v}) = c_i$.

There is a tricky point which will come up in Lemma 4 below. We only really care about the germ of $\xi_j$ at $S_j$ and as we take smaller and smaller neighborhoods $U_j$, the set $\mathbf{B}_i(\mathbf{v})$ can get smaller and thus some virtual zeroes disappear. We could give a more complicated definition of virtual zero to allow this, we do this below. But for the sake of simplicity in the proofs and attempting to keep technical details from obscuring the main ideas we prefer to use the above simpler definition. It just sometimes identifies a point as a virtual zero when it is not necessary to do so.

Following is the more complicated definition of virtual zero, which gives fewer virtual zeroes in general, so for example it is easier to obtain isolated virtual zeroes. Most readers should ignore this paragraph, at least on first reading. We presume one could also say a point is a virtual zero if it is a virtual zero in the above sense for all germ representatives, but haven't actually checked this. Let $i_1 < i_2 < \cdots < i_k$ be indices of skeleta of $X$. We let $B_{i_1 i_2 \cdots i_k}$ be the set of $x \in \bigcap_{j=1}^{k} U_{i_j} - X_{i_k}$ so that for some numbers $c_j < 0$

$$\mathbf{v}(x) = \sum_{j=1}^{k} c_j \xi_{i_j}(x).$$

We let

$$C^k_{i_1 i_2 \cdots i_k} = S_{i_k} \cap \mathrm{Cl}(B_{i_1 i_2 \cdots i_k})$$

and inductively define

$$C^{j}_{i_1 i_2 \cdots i_k} = S_{i_j} \cap \mathrm{Cl}(C^{j+1}_{i_1 i_2 \cdots i_k})$$

for $k > j \geq 1$. Then as an alternative we could say $x$ is a virtual zero of $\mathbf{v}$ if $x \in C^{1}_{i_1 i_2 \cdots i_k}$ for some $i_1 < \cdots < i_k$. Note that if $x$ is not a virtual zero in this complicated sense, then after shrinking the $U_i$'s sufficiently $x$ is not a virtual zero in the simpler sense. To see this, pick a compact neighborhood $N$ of $x$ so that $N \cap \mathrm{Cl}(C^{2}_{i_1 i_2 \cdots i_k})$ is empty. Then after deleting a neighborhood of $N \cap \mathrm{Cl}(C^{3}_{i_1 i_2 \cdots i_k})$ from $U_{i_2}$ we have $N \cap \mathrm{Cl}(C^{3}_{i_1 i_2 \cdots i_k}) = \emptyset$. Then we may delete a neighborhood of $N \cap \mathrm{Cl}(C^{4}_{i_1 i_2 \cdots i_k})$ from $U_{i_3}$ to get $N \cap \mathrm{Cl}(C^{4}_{i_1 i_2 \cdots i_k}) = \emptyset$. Continuing this, we eventually get $N \cap B_{i_1 i_2 \cdots i_k} = \emptyset$. After doing this for all multi-indices you get $x \notin \mathrm{Cl}(\mathbf{B}_{i_1}(\mathbf{v}))$.

We say a vector field $\mathbf{v}$ is **semiradial** if it has no virtual zeroes.

For example the figure below shows a semiradial vector field. The center line is a stratum. The vector field is not radial since it sometimes slopes toward the center line. But it is semiradial since it never points straight up or down towards the center line.

We say that a vector field $\mathbf{v}'$ on $\partial X$ is a **semiradial projection** of $\mathbf{v}$ to $\partial X$ if it is semiradial, has isolated zeroes and also for each $x \in \partial X$ so that $\mathbf{v}(x)$ is tangent to $\partial X$ we know that $\mathbf{v}'(x) = \mathbf{v}(x)$.

The following lemma shows the existence of semiradial projections. There are examples which show that it is necessary to shrink the $U_i$'s if the simple definition of virtual zero is used. It is not necessary to do this if the more complicated definition is used.

**Lemma 4** *Suppose $\mathbf{v}$ is a vector field on $X$ with no zeroes or virtual zeroes on $\partial X$. Then (after perhaps shrinking the $U_i$'s) there is a vector field $\mathbf{v}'$ on $\partial X$ which is a semiradial projection of $\mathbf{v}$ to $\partial X$. Furthermore, if $\mathbf{v}$ is radial we may take $\mathbf{v}'$ to be radial.*

**Proof:** Let $T \subset \partial X$ be the set of $x$ at which $\mathbf{v}(x)$ is tangent to $\partial X$. Suppose by induction on $k$ that we have a vector field $\mathbf{v}''$ on $\partial X$ which equals $\mathbf{v}$ on $T$ and has no virtual zeroes in $X_{k-1}$. For every $i \geq k$, let $U'_i$ be a neighborhood of $S_i$ in $X$ so that $U'_i \subset U_i - \mathbf{B}_i(\mathbf{v}) \cap \partial X$ and so $U_i \cap S_j$ is closed in $S_j$ for all $j > i$. Set $\xi'_i = \xi_i$ on $U'_i$ and let $\xi'_i$ be 0 outside $U'_i$. We let $\mathbf{B}'_i(\mathbf{v}'')$ be the subset of $\mathbf{B}_i(\mathbf{v}'') \cap U'_i$ so that for some numbers $c_j \leq 0$ for $i \leq j \leq m$ we have

$$\mathbf{v}''(x) = \sum_{j=i}^{m} c_j \xi'_j(x)$$

and $c_i < 0$. In other words, $\mathbf{B}'_i$ is just like $\mathbf{B}_i$ except we shrink the neighborhoods $U_i$ to $U'_i$. Note that $B'_i(\mathbf{v}) \cap \partial X = \emptyset$. Now

$$\mathrm{Cl}(\mathbf{B}'_k(\mathbf{v}'') \cap S_i) \cap S_i \cap T \subset \bigcup_{j \geq k} \mathbf{B}'_j(\mathbf{v}'') \cap S_i \cap T = \bigcup_{j \geq k} \mathbf{B}'_j(\mathbf{v}) \cap S_i \cap T = \emptyset.$$

So using the Tietze extension theorem we may find a continuous function $\eta_i: S_i \to [0,\infty)$ which is 0 on $S_i - U_k$ and on $T \cap S_i$ and so that $\eta_i(x) > -d\rho_k(\mathbf{v}''(x))$ for all $x \in \mathbf{B}'_k(\mathbf{v}'') \cap S_i$. Define $\eta: X \to [0,\infty)$ to be 0 on $X_k$ and $\eta_i$ on $S_i$ for $i > k$. Now shrink each $U_i$ until it is contained in $U'_i$. After shrinking, $\mathbf{B}_k(\mathbf{v}'') \subset \mathbf{B}'_k(\mathbf{v}'')$. Then the vector field $\mathbf{v}'' + \eta\xi_k$ has no virtual zeroes in $X_k$ since $\mathbf{B}_i(\mathbf{v}'' + \eta\xi_k) \subset \mathbf{B}_i(\mathbf{v}'')$ for $i \leq k$ and $d\rho_k(\mathbf{v}'' + \eta\xi_k) > 0$ on $\mathbf{B}_k(\mathbf{v}'')$. Thus we may by induction assume that $\mathbf{v}''$ is semiradial.

If $\mathbf{v}$ is radial, then we may modify the above construction by asking that $\eta(x) > -d\rho_k(\mathbf{v}''(x))$ for all $x$ near $S_k$. This is possible since $d\rho_k(\mathbf{v}''(x)) = d\rho_k(\mathbf{v}(x)) > 0$ for all $x \in T$ near $S_k$. Then the resulting $\mathbf{v}''$ would be radial.

So we may by induction assume such a semiradial or radial $\mathbf{v}''$ exists. Now to construct $\mathbf{v}'$ we perturb $\mathbf{v}''$ to make its zeroes isolated, just as we did in the proof of Lemma 1.

**Q.E.D.**

**Lemma 5** *Suppose $\mathbf{v}$ is a semiradial vector field on $X$ with isolated zeroes and no zeroes on $\partial X$. Then there is a radial vector field $\mathbf{v}'$ on $X$ so that:*

1) *$\mathbf{v}'$ has the same zeroes as $\mathbf{v}$ and in fact $\mathcal{I}(p,\mathbf{v}) = \mathcal{I}(p,\mathbf{v}')$ at all points $p \in X$.*
2) *$\mathbf{v}'$ is tangent to $\partial X$ at the same places as $\mathbf{v}$ is tangent to $\partial X$.*
3) *$\mathbf{v}'$ points inward on $\partial X$ at the same places as $\mathbf{v}$.*
4) *$\mathbf{v}'$ points outward on $\partial X$ at the same places as $\mathbf{v}$.*
5) *If $\mathbf{v}''$ is a semiradial projection of $\mathbf{v}$ to $\partial X$, there is a radial projection $\mathbf{v}'''$ of $\mathbf{v}'$ to $\partial X$ so that $\mathbf{v}'''$ and $\mathbf{v}''$ have the same zeroes and in fact $\mathcal{I}(p,\mathbf{v}'') = \mathcal{I}(p,\mathbf{v}''')$ at all points $p \in \partial X$.*

**Proof:** Pick neighborhoods $U''_i$ of $S_i$ in $U_i$ so that $U''_i \cap \mathbf{B}_i(\mathbf{v})$ and $U''_i \cap \mathbf{B}_i(\mathbf{v}'')$ are empty and if $p \in U''_i$ is a zero of $\mathbf{v}$ then $p \in S_i$ and if $p \in U''_i \cap \partial X$ is a zero of $\mathbf{v}''$ then $p \in S_i$. Let $T \subset \partial X$ be the set of $x$ at which $\mathbf{v}(x)$ is tangent to $\partial X$.

The proof is similar to the proof of Lemma 1 but we must exercise more care. We will construct $\mathbf{v}'$ and $\mathbf{v}'''$ by letting $\mathbf{v}' = \mathbf{v} + \sum_{i=0}^{m-1} \beta_i\,\xi_i$ and $\mathbf{v}''' = \mathbf{v}'' + \sum_{i=0}^{m-1} \beta_i\,\xi_i$ for suitably chosen nonnegative functions $\beta_i$. Then conditions 2), 3) and 4) are automatically satisfied.

We suppose by induction on $k$ that we have constructed nonnegative functions $\beta_i: X \to [0,\infty)$ for $0 \leq i < k$ so that:

a) $\beta_i$ is continuous on each stratum.
b) $\beta_i$ vanishes on $X_i$ and $X - U''_i$ and in fact for each $j > i$ the support of the restriction $\beta_i|_{S_j}$ is contained in $U''_i \cap S_j$.
c) For each $j < k$, there is a neighborhood $U'_j$ of $S_j$ in $U''_i$ so that $d\rho_j(\mathbf{v} + \sum_{i=0}^{k-1} \beta_i\,\xi_i) > 0$ on $U'_j - S_j$.
d) For each $j < k$ we have $d\rho_j(\mathbf{v}'' + \sum_{i=0}^{k-1} \beta_i\,\xi_i) > 0$ on $U'_j \cap \partial X - S_j$.

To complete the induction, we just choose $\beta_k$ continuous on each stratum, vanishing on $X_k$, with support in $U''_k \cup X_k$, thus giving a) and b), so that for some neighborhood $U'_k$ of $S_k$ we have

$$\beta_k(x) > -d\rho_k(\mathbf{v}(x)) - \sum_{i=0}^{k-1} \beta_i(x)\, d\rho_k(\xi_i(x))$$

for all $x \in U'_k - S_k$ and so

$$\beta_k(x) > -d\rho_k(\mathbf{v}''(x)) - \sum_{i=0}^{k-1} \beta_i(x)\, d\rho_k(\xi_i(x))$$

for all $x \in \partial X \cap U'_k - S_k$.

So by induction we may assume that $k = m$ and so we have $\beta_1, \ldots, \beta_{m-1}$ satisfying a)-d) above. We let $\mathbf{v}' = \mathbf{v} + \sum_{i=0}^{m-1} \beta_i\, \xi_i$. The vector field $\mathbf{v}'$ is radial by c). Likewise, $\mathbf{v}''' = \mathbf{v}'' + \sum_{i=0}^{m-1} \beta_i\, \xi_i$ is radial by d).

Let us see why 1) is true. First of all, if $x$ is a zero of $\mathbf{v}$ in some stratum $S_k$ then $x \notin U''_i$ for all $i < k$ so $x$ is not in the support of $\beta_i|_{S_k}$ for all $i$ so $\mathbf{v}$ and $\mathbf{v}'$ agree on a neighborhood of $x$ in $S_k$ and thus $\mathcal{I}(x, \mathbf{v}) = \mathcal{I}(x, \mathbf{v}')$. So we must only show that $\mathbf{v}'$ has no zeroes which are not zeroes of $\mathbf{v}$. Suppose $\mathbf{v}'(x) = 0$ but $\mathbf{v}(x) \neq 0$. Pick the smallest $\ell$ so that $\beta_\ell(x) \neq 0$. Then $x \in \mathbf{B}_\ell(\mathbf{v})$ since

$$\mathbf{v}(x) = \sum_{j=\ell}^{m-1} -\beta_j(x)\xi_j(x)$$

and $-\beta_\ell(x) < 0$. This is a contradiction, since $x \in \mathbf{B}_\ell(\mathbf{v}) \cap U''_\ell$ which is empty. So 1) is true. By a similar argument, 5) is true. **Q.E.D.**

We can now generalize Theorem 3 to semiradial vector fields.

**Corollary 6** *Suppose $\mathbf{v}$ is a semiradial vector field on a compact weak mapping cylindered manifold complex with boundary $X$, the zeroes of $\mathbf{v}$ are isolated and there are no zeroes on the boundary of $X$. Let $\mathbf{v}'$ be a semiradial projection of $\mathbf{v}$ to $\partial X$. Then*

$$\chi(X) = \sum_{p \in X} \mathcal{I}(p, \mathbf{v}) + \sum_{q \in A} \mathcal{I}(q, \mathbf{v}')$$

*where $A$ is the set of points $x \in \partial X$ such that $\mathbf{v}(x)$ points into $X$.*

**Proof:** By Lemma 5, we may find a radial vector field $\mathbf{v}''$ on $X$ with the same zeroes as $\mathbf{v}$ and a radial projection $\mathbf{v}'''$ of $\mathbf{v}''$ to $\partial X$ so that $\mathbf{v}'''$ and $\mathbf{v}'$ have the same zeroes. Moreover, $\mathcal{I}(p, \mathbf{v}) = \mathcal{I}(p, \mathbf{v}'')$ for all $p \in X$ and $\mathcal{I}(p, \mathbf{v}''') = \mathcal{I}(p, \mathbf{v}')$ for all $p \in \partial X$. Also, $\mathbf{v}$ and $\mathbf{v}''$ point outward on the same subset $A \subset \partial X$. By Theorem 3 we know that

$$\chi(X) = \sum_{p \in X} \mathcal{I}(p, \mathbf{v}'') + \sum_{q \in A} \mathcal{I}(q, \mathbf{v}''').$$

Hence we have

$$\chi(X) = \sum_{p \in X} \mathcal{I}(p, \mathbf{v}) + \sum_{q \in A} \mathcal{I}(q, \mathbf{v}').$$

**Q.E.D.**

## A Poincaré-Hopf theorem in the presence of virtual zeroes

Now what if a vector field $\mathbf{v}$ is not semiradial, so it has virtual zeroes. It is not hard to find examples where the conclusion of Corollary 6 is not true. But we show that if the index is suitably defined at a zero or virtual zero, then a Poincaré-Hopf theorem does hold.

For example, let $Y$ and $Z$ be compact smooth manifolds without boundary and let $f: Y \to Z$ be a smooth map. We let $X$ be the mapping cylinder of $f$, the quotient space $Y \times [0,1] \cup Z/(y,0) \sim f(y)$. We stratify $X$ by letting $S_0 = Z$ and $S_1 = Y \times (0,1]$. We have $\partial X = Y \times 1$. We let $U_0 = X - \partial X$, $L_0 = Y$, $q_0(y,t) = (y,t)$, $\pi_0(y,t) = f(y)$ and $\rho_0(y,t) = t$. Thus $X$ is a mapping cylindered manifold complex with boundary. Consider a vector field $\mathbf{v}$ on $X$ with isolated zeroes so that on $Y \times (0,1]$ we have $\mathbf{v}(y,t) = (\mathbf{v}'(y), -1)$ for some vector field $\mathbf{v}'$ on $Y$ with isolated zeroes. Let $K \subset Y$ be the zeroes of $\mathbf{v}'$. Then $\mathbf{B}_0(\mathbf{v}) = K \times (0,1)$ so $f(K)$ is the set of virtual zeroes of $\mathbf{v}$. Note

$$\sum_{p \in X} \mathcal{I}(p, \mathbf{v}) = \sum_{p \in Z} \mathcal{I}(p, \mathbf{v}) = \chi(Z) = \chi(X).$$

But $\mathbf{v}$ points inward on $\partial X$ so we want this sum to be $\chi(X) - \chi(\partial X) = \chi(X) - \chi(Y)$. We will show below how to suitably define an index at the virtual zeroes of $\mathbf{v}$ so that the sum does come out right. In this example, the modified index at a virtual zero $p$ will turn out to be $\mathcal{I}(p, \mathbf{v}) - \sum_{q \in f^{-1}(p) \cap K} \mathcal{I}(q, \mathbf{v}')$.

A **nice contractible neighborhood** of a point $x \in X$ is a compact neighborhood $W$ of $x$ so that $W$ is contractible and so that for each stratum $S_i$, $W \cap S_i$ is a smooth codimension zero submanifold of $S_i$ and the germ of $W$ at $W \cap S_i$ is the germ of $\pi_i^{-1}(W \cap S_i)$ at $W \cap S_i$ and the germ of the frontier $\mathrm{Fr}(W)$ at $\partial(W \cap S_i)$ is the germ of $\pi_i^{-1}(\partial(W \cap S_i))$.

Note that $W$ and the complement of the interior of $W$ each have a natural structure as a mapping cylindered manifold complex with boundary since the functions $\pi_i$, $\rho_i$ and $q_i$ will just restrict after perhaps shrinking the $U_i$'s and $V_i$'s. If $x \notin \partial X$ then $\partial W = \mathrm{Fr}(W)$.

We say a mapping cylindered manifold complex with boundary $X$ is **nice** if the projection functions $\pi_i$ are smooth when restricted to each stratum, $d\pi_i(\xi_i) = 0$, and for all $i < j$ we have $\pi_i \pi_j = \pi_i$ on $U_i \cap \pi_j^{-1}(U_i)$.

For example a Whitney stratified set is nice.

**Lemma 7** *Let $X$ be a nice mapping cylindered manifold complex with boundary. Then every point $x \in X - \partial X$ has arbitrarily small nice contractible neighborhoods.*

**Proof:** Let $i$ be such that $x \in S_i$. Pick any open neighborhood $U$ of $x$ in $U_i$. Pick a smooth function $r: S_i \to [0, \infty)$ so that $x = r^{-1}(0)$ and so that $r^{-1}([0,\epsilon])$ is contractible for all small enough $\epsilon$, for example the squared distance function in some Riemannian metric. By Sard's theorem we may pick a small $\epsilon > 0$ so that for each $j \geq i$, $\epsilon$ is a regular value of the restriction of $r \circ \pi_i$ to $U_i \cap S_j$. Also we may suppose that $r^{-1}([0,\epsilon]) \subset U$. Pick $\delta > 0$ so that $\rho_i^{-1}([0,\delta]) \cap \pi_i^{-1} r^{-1}([0,\epsilon]) \subset U$. Then we may let $W$ be $\rho_i^{-1}([0,\delta]) \cap \pi_i^{-1} r^{-1}([0,\epsilon])$ with corners rounded. To be more precise about corner rounding, note that since $\pi_i q_i(y,t) = q_i(y,0)$ is independent of $t$, any critical value of $r\pi_i$ is a critical value of the map $y \mapsto r q_i(y,0)$. By properness of $q_i$, the set of critical values of $r\pi_i$ must then be closed. Consequently, there is an $\epsilon' \in (0,\epsilon)$ so that $r\pi_i$ has no critical values in $[\epsilon', \epsilon]$. Pick a continuous function $\gamma\colon [0,\epsilon] \to (\delta/2, \delta]$ so that $\gamma(t) = \delta$ if $t \le \epsilon'$ and $\{ (x,y) \in (0,\epsilon] \times (0,\delta] \mid y \le \gamma(x) \}$ is a smooth submanifold of $(0, 2\epsilon] \times (0, 2\delta]$. Then let $W = \{ x \in U_i \mid r\pi_i(x) \le \epsilon \text{ and } \rho_i(x) \le \gamma r \pi_i(x) \}$. We see that $W$ is contractible by noting that it deformation retracts to $r^{-1}([0,\epsilon])$ which is contractible. **Q.E.D.**

We now define the index of a vector field $\mathbf{v}$ at an isolated zero or virtual zero $x$ of $\mathbf{v}$ if $x \notin \partial X$ and $x$ has arbitrarily small nice contractible neighborhoods. Pick a nice contractible neighborhood $W$ of $x$ so that $x$ is the only zero or virtual zero of $\mathbf{v}$ in $W$. Choose any vector field $\mathbf{v}'$ on $\partial W$ which is a semiradial projection of $\mathbf{v}|_W$ to $\partial W$. Then we define

$$\hat{\mathcal{I}}(x, \mathbf{v}) = 1 - \sum_{q \in A} \mathcal{I}(q, \mathbf{v}')$$

where $A$ is the set of $q \in \partial W$ so that $\mathbf{v}(q)$ points into $W$. If $x$ is not a zero or virtual zero of $\mathbf{v}$ then we define $\hat{\mathcal{I}}(x, \mathbf{v}) = 0$.

**Lemma 8** *$\hat{\mathcal{I}}(x, \mathbf{v})$ is independent of the choice of $W$ and the projection $\mathbf{v}'$.*

**Proof:** Pick another nice contractible neighborhood $W'$ with $W'$ contained in the interior $\operatorname{Int} W$ of $W$ and pick a semiradial projection $\mathbf{v}''$ of $\mathbf{v}|_{W'}$ to $\partial W'$. Let $Z = W - \operatorname{Int} W'$. By Corollary 6 we know that

$$\chi(Z) = \sum_{q \in A} \mathcal{I}(q, \mathbf{v}') + \sum_{q \in C} \mathcal{I}(q, \mathbf{v}'')$$

where $A$ is the set of $q \in \partial W$ so that $\mathbf{v}(q)$ points into $W$ and $C$ is the set of $q \in \partial W'$ so that $\mathbf{v}(q)$ points out of $W'$, i.e., into $Z$. We also know that

$$1 = \chi(W) = \chi(Z) + \chi(W') - \chi(\partial W') = \chi(Z) + 1 - \chi(\partial W')$$

so $\chi(Z) = \chi(\partial W')$. But $\chi(\partial W') = \sum_{q \in \partial W'} \mathcal{I}(q, \mathbf{v}'')$ so we get

$$\sum_{q \in A} \mathcal{I}(q, \mathbf{v}') = \sum_{q \in \partial W' - C} \mathcal{I}(q, \mathbf{v}'').$$

Calculating the index using $W$ and $\mathbf{v}'$ we obtain

$$\hat{\mathcal{I}}(x, \mathbf{v}) = 1 - \sum_{q \in A} \mathcal{I}(q, \mathbf{v}').$$

Calculating using $W'$ and $\mathbf{v}''$ we obtain

$$\hat{\mathcal{I}}(x, \mathbf{v}) = 1 - \sum_{q \in A'} \mathcal{I}(q, \mathbf{v}'')$$

where $A'$ is the set of points on $\partial W'$ where $\mathbf{v}$ points into $W'$. But $\mathcal{I}(q, \mathbf{v}'') = 0$ if $q \in \partial W' - (A' \cup C)$ since $\mathbf{v}''(q) \neq 0$ (note $\mathbf{v}(q)$ is tangent to $\partial W'$). So

$$\sum_{q \in \partial W' - C} \mathcal{I}(q, \mathbf{v}'') = \sum_{q \in A'} \mathcal{I}(q, \mathbf{v}'')$$

and the result follows. **Q.E.D.**

**Lemma 9** *If* $\mathbf{v}$ *is semiradial and has isolated zeroes, then* $\hat{\mathcal{I}}(x, \mathbf{v}) = \mathcal{I}(x, \mathbf{v})$ *for all* $x \in X$.

**Proof:** Both quantities are 0 if $x$ is not a zero of $\mathbf{v}$, so assume that $x$ is an isolated zero of $\mathbf{v}$. Let $W$ be a nice contractible neighborhood of $x$ which contains no other zeroes of $\mathbf{v}$. By Corollary 6, we know that

$$\chi(W) = \sum_{p \in W} \mathcal{I}(p, \mathbf{v}) + \sum_{q \in A} \mathcal{I}(q, \mathbf{v}').$$

But $W$ is contractible so $\chi(W) = 1$. Also $\mathcal{I}(p, \mathbf{v}) = 0$ if $p \in W - x$ since $\mathbf{v}(p) \neq 0$ for $p \in W - x$. So we get

$$\mathcal{I}(x, \mathbf{v}) = 1 - \sum_{q \in A} \mathcal{I}(q, \mathbf{v}') = \hat{\mathcal{I}}(x, \mathbf{v}).$$

**Q.E.D.**

**Theorem 10** *Let* $X$ *be a nice compact weak mapping cylindered manifold complex with boundary. Let* $\mathbf{v}$ *be a vector field on* $X$ *with isolated zeroes and isolated virtual zeroes and with no zeroes or virtual zeroes on* $\partial X$. *Let* $\mathbf{v}'$ *be a semiradial projection of* $\mathbf{v}$ *to* $\partial X$. *Let* $A$ *be the set of points in* $\partial X$ *where* $\mathbf{v}(x)$ *points into* $X$. *Then*

$$\chi(X) = \sum_{p \in X} \hat{\mathcal{I}}(p, \mathbf{v}) + \sum_{q \in A} \mathcal{I}(q, \mathbf{v}').$$

**Proof:** We prove this by induction on the number of virtual zeroes of $\mathbf{v}$. If $\mathbf{v}$ has no virtual zeroes, then this follows from Corollary 6 and Lemma 9.

Now suppose that $\mathbf{v}$ has a virtual zero at some $x$. Pick a nice contractible neighborhood $W$ of $x$ which contains no zeroes or virtual zeroes other than $x$. Let $X' = X - \mathrm{Int}(W)$. Pick a semiradial projection $\mathbf{v}''$ of $\mathbf{v}|_W$ to $\partial W$ and let $B$ be the set of points of $\partial W$ at which $\mathbf{v}$ points out of $W$, and hence into $X'$. By induction on the number of virtual zeroes we know that

$$\chi(X') = \sum_{p \in X'} \hat{\mathcal{I}}(p, \mathbf{v}) + \sum_{q \in A} \mathcal{I}(q, \mathbf{v}') + \sum_{q \in B} \mathcal{I}(q, \mathbf{v}'').$$

So we only need show that

$$\chi(X) - \chi(X') = \hat{\mathcal{I}}(x, \mathbf{v}) - \sum_{q \in B} \mathcal{I}(q, \mathbf{v}'').$$

But

$$\chi(X) - \chi(X') = \chi(W) - \chi(\partial W) = 1 - \chi(\partial W)$$

so the result follows since $\chi(\partial W) = \sum_{q \in \partial W} \mathcal{I}(q, \mathbf{v}'')$ and $\hat{\mathcal{I}}(x, \mathbf{v}) = 1 - \sum_{q \in \partial W - B} \mathcal{I}(q, \mathbf{v}'')$.
**Q.E.D.**

Here are two simple examples which may be helpful to the reader. First let $X$ be the closed unit disc $D^2$ in the plane, stratified by the unit circle $S^1$ and the interior $\mathrm{Int}(D^2)$ of the disc. (So $\partial X = \emptyset$.) Let $v$ be $\frac{\partial}{\partial\theta}$ on $S^1$ and $\frac{\partial}{\partial x}$ on $\mathrm{Int}(D^2)$. Then $v$ has no zeros, is not continuous, and is not radial or semiradial, but has a single virtual zero at $(0,1)$ if we use the standard mapping cylinder structure near $S^1$, considered as a stratum. It is now an easy but instructive exercise to check that the virtual index of $v$ at $(0,1)$ is 1.

Next we give an example with a nonempty boundary : let $S_2$ be the unit 2-sphere with the north and south poles removed, and let $S_1$ be the closed line segment with endpoints $(0,0,2)$ and $(0,0,-2)$. Then set $X = S_1 \bigcup S_2$. Consider the stratified vector field $v$ on $X$ which equals $\frac{\partial}{\partial z}$ on $S_1$, and on $S_2$ is the orthogonal projection of $\frac{\partial}{\partial z}$ onto the tangent plane at each point of $S_2$. Then $v$ has no zeros, again is not continuous, nor radial, and has just one virtual zero at the north pole $(0,0,1)$, of virtual index 0. The formula of Theorem 10 works because of the boundary contribution : $v$ is entering on $\partial X$ at $(0,0,-2)$, and the associated semiradial projection at this point, which is zero, has an index of $+1$, which is the Euler characteristic of $X$.

Now we indicate situations in which Theorem 10 applies to "generic" vector fields. Let $X$ be a mapping cylinder manifold complex with boundary embedded in a smooth manifold $M$ such that the strata are Whitney (a)-regular. Let $\mathbf{v}_M$ be a vector field on $M$, and $\mathbf{v}$ the vector field on $X$ obtained by projecting $\mathbf{v}_M$ onto the tangent space of the stratum of $X$ at each point. Then every virtual zero of $\mathbf{v}$ is a zero. As a corollary, if $M$ is some $\mathbf{R^n}$, then the set of $\mathbf{v}_M$ such that $\mathbf{v}$ has only isolated zeroes (and virtual zeroes) none of which lie on $\partial X$, is open and dense in the space of constant vector fields on $\mathbf{R^n}$. Hence Theorem 10 applies to such "generic" vector fields. As another corollary, suppose $X$ is a weak mapping cylindered manifold complex with boundary embedded in a manifold $M$ and $f\colon M \to \mathbf{R}$ is a smooth function so that $f|_X$ has only isolated critical points on the strata of $X$. (This is a generic condition.) Then Theorem 10 applies to the gradient vector field of $f|_{S_i}$.

There is another point of view of vector fields on mapping cylindered manifold complexes with boundary. We can look at a resolution of singularities of a weak mapping cylindered manifold complex with boundary to a disjoint union of manifolds $V_i$ with boundary and gluing maps which glue parts of the boundary of some of these manifolds to others. Thom looked at a version of this in his paper **[T]**. Alternatively, this is similar to the resolution towers of **[AK2]** except that one splits the manifolds apart along the ticos to obtain a manifold with boundary. In other words, we obtain each skeleton $X_i$ from $X_{i-1}$ by attaching a manifold $V_i$ along a reasonably nice closed subset of its boundary. It is easy to prove, by induction, the formula

$$\chi(X) - \chi(\partial X) = \sum_{i=1}^{n} \chi(V_i) - \chi(\partial V_i).$$

Thus Poincaré-Hopf theorems for mapping cylindered manifold complexes with boundary can be reduced to the manifold case. This also explains why we only needed our vector fields to be continuous on each stratum, since the Euler characteristic only depends on the resolutions $V_i$ of each stratum. Our virtual zeroes correspond to possible zeroes of a projection of an extension of the vector field on $\operatorname{Int} V_i$ to $\partial V_i$. Such a continuous extension might not exist, but will after perturbation.

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