Real Algebraic Interiors and a Problem of Arnol'd

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1. Introduction

We give a necessary and sufficient topological condition for a smooth, compact manifold with boundary to have a codimension 1 real algebraic interior. Theorem 1 states that if X^n is a smooth, compact manifold with nonempty boundary, then there is a smooth, proper embedding $X^n \hookrightarrow D^{n+1}$ if and only if the interior of X^n is diffeomorphic to a nonsingular, real algebraic subset of $\mathbb{R}^{n+1}$. Theorem 1 is a noncompact version of a classical theorem of Seifert (Theorem 4 of [**S**]).

V.I. Arnol'd has asked whether there is an exotic $\mathbb{R}^4$ that is a nonsingular, real algebraic subset of $\mathbb{R}^5$ (related is [**A**]). The impetus for the work presented here is the remarkable result of Meersseman and Verjovsky: *if Arnol'd's problem has an affirmative answer then either the classical 3-dimensional Poincaré Conjecture is false or the smooth 4-dimensional Poincaré Conjecture is false* [**MV**]. In a similar vein, if there is an exotic $\mathbb{R}^4$ that is the interior of a smooth, compact manifold with boundary, then either the classical 3-dimensional Poincaré Conjecture is false or the smooth 4-dimensional Poincaré Conjecture is false ([**GS**], p.366). As an application of Theorem 1, we show (Corollary 1) that, in fact, there exists an exotic $\mathbb{R}^4$ that is a nonsingular, real algebraic subset of $\mathbb{R}^5$ if and only if there is an exotic $\mathbb{R}^4$ that is the interior of a smooth, compact manifold with boundary.

2. Results

Our main result is:

THEOREM 1. *Let X^n be a smooth, compact manifold with nonempty boundary. Then int (X^n) is diffeomorphic to a nonsingular real algebraic subset of $\mathbb{R}^{n+1}$ if and only if X^n admits $X^n \hookrightarrow D^{n+1}$ a smooth, proper embedding. If such an embedding exists, then int (X^n) is properly isotopic to a nonsingular real algebraic subset W of int $\left(D^{n+1}\right) \approx \mathbb{R}^{n+1}$. In fact, for all balls B_R^{n+1} of sufficiently large radius R, the pair $(B_R^{n+1}, B_R^{n+1} \cap W)$ is diffeomorphic to (D^{n+1}, X^n).*

REMARK 1. *A proper map $X^n \hookrightarrow D^{n+1}$ is assumed to be transverse at ∂D^{n+1} and ∂X^n maps to ∂D^{n+1}.*

We apply Theorem 1 to a problem of V.I. Arnol'd that is related to that in [**A**].

PROBLEM 1 (Arnol'd). *Does there exist an exotic $\mathbb{R}^4$ diffeomorphic to a nonsingular, real algebraic subset of $\mathbb{R}^5$?*

REMARK 2. *By 'exotic' we always mean a smooth manifold that is homeomorphic but not diffeomorphic to the 'standard' one. An n-dimensional pseudodisk is a smooth compact contractible n-manifold with boundary.*

In [**MV**], it was shown that any exotic $\mathbb{R}^4$, $\mathcal{R}$, answering Arnol'd's problem in the affirmative is necessarily collared at infinity by a smooth homotopy $3-$sphere (possibly S^3) that smoothly embeds in S^4. Thus, if such an $\mathcal{R}$ exists, then either the 3-dimensional or smooth 4-dimensional Poincaré Conjecture is false. As Gompf and Stipsicz point out [**GS**], p.366, the same result follows from the existence of an exotic $\mathbb{R}^4$ as the interior of a smooth compact manifold with boundary. This suggests there is a similarity in structure between algebraic exotic $\mathbb{R}^4 s$ in $\mathbb{R}^5$ and exotic $\mathbb{R}^4 s$ as the interiors of compact manifolds. This is, in fact, the case. We will need the following observation.

LEMMA 1. *Let X^4 be a pseudodisk with simply connected boundary. Then, X^4 admits $X^4 \hookrightarrow D^5$ a smooth proper embedding if and only if ∂X^4 smoothly embeds in S^4.*

PROOF. One direction is obvious, so assume there is $\partial X^4 \hookrightarrow S^4$ a smooth embedding. Let Y^5 be the smooth manifold with boundary obtained from $S^4 \times [0,1]$ by gluing on $X^4 \times [-1,1]$ along a product neighborhood, $N = \partial X^4 \times [-1,1] \subset S^4 \times 0$, of ∂X^4 in the canonical way and smoothing corners. Then, ∂Y^5 consists of three connected components, say $S^4 \times 1$, Σ^4_A and Σ^4_B. Recalling that ∂X^4 is simply connected, standard theorems imply $\Sigma^4_{A,B}$ are simply connected $\mathbb{Z}$-homology 4-spheres, hence, topological 4-spheres by Freedman's theorem [**F**]. Now, the 4^{th} homotopy sphere cobordism group is trivial, i.e. $\Theta_4 = 0$, [**KM**]. Thus, there exists $W^5_{A,B}$ smooth null h-cobordisms of $\Sigma^4_{A,B}$ respectively (sew a standard 5-disk onto a smooth h-cobordism between $\Sigma^4_{A,B}$ and S^4). Let Z^5 be the smooth manifold with boundary obtained from Y^5 by sewing on $W^5_{A,B}$ along $\Sigma^4_{A,B}$ in the canonical way. Again, standard thoerems imply Z^5 is simply connected and has the integral homology of a point. As $\partial Z^5 = S^4$, we may conclude that Z^5 is diffeomorphic to D^5 [**M2**], p.110. The result follows. □

Combining this with Theorem 1 we get:

COROLLARY 1. *There exists an exotic $\mathbb{R}^4$ diffeomorphic to a nonsingular, real algebraic subset of $\mathbb{R}^5$ if and only if there is an exotic $\mathbb{R}^4$ diffeomorphic to the interior of a smooth, compact manifold with boundary.*

PROOF. First, suppose $\mathcal{R}$ is an exotic $\mathbb{R}^4$ that is a nonsingular, real algebraic subset of $\mathbb{R}^5$. By [**MV**], there is a homotopy 3-sphere Σ^3 and a neighborhood of infinity in $\mathcal{R}$ that is diffeomorphic to $\Sigma^3 \times [0,1)$. Let X^4 be the compact manifold obtained from $\mathcal{R}$ by removing $\Sigma^3 \times (0,1)$, so $int\left(X^4\right)$ is diffeomorphic to $\mathcal{R}$. The result follows.

For the other direction, suppose X^4 is a smooth, compact manifold with boundary and $int\left(X^4\right)$ is diffeomorphic to $\mathcal{R}$, an exotic $\mathbb{R}^4$. Reparameterizing collars, we see that X^4 is homotopy equivalent to $\mathcal{R}$, and so X^4 is contractible. Also, ∂X^4 is a homotopy 3-sphere [**GS**], pp.366 and 519. Therefore, X^4 is a pseudodisk with simply connected boundary. There are two cases:

Case 1. ∂X^4 smoothly embeds in S^4. Then, Lemma 1 and Theorem 1 imply $\mathcal{R}$ is diffeomorphic to a real algebraic subset of $\mathbb{R}^5$.

Case 2. ∂X^4 does not smoothly embed in S^4. The punctured double, $2X^4 - pt$, is a smooth manifold homeomorphic to $\mathbb{R}^4$ since the double, $2X^4$, of X^4 is homeomorphic to S^4 by [**F**]. However, $2X^4 - pt$ is not diffeomorphic to $\mathbb{R}^4$ since otherwise we have a smooth embedding of ∂X^4 in S^4. Thus, $2X^4 - pt$ is an exotic $\mathbb{R}^4$ collared at infinity by S^3. Lemma 1 and Theorem 1 imply $2X^4 - pt$ is diffeomorphic to a real algebraic subset of $\mathbb{R}^5$. $\square$

Thus, Arnol'd's real algebraic problem is equivalent to a topological one. We remind the reader that all exotic $\mathbb{R}^4 s$, $\mathcal{R}$, are smooth proper submanifolds of $\mathbb{R}^5$ since $\mathcal{R} \times \mathbb{R} \approx \mathbb{R}^5$ either by engulfing or the smooth proper h-cobordism theorem. Moreover, these smooth embeddings may be chosen to be real analytic. Still, there are only countably many smooth compact manifolds (with or without boundary) and there are uncountably many pairwise nondiffeomorphic exotic $\mathbb{R}^4 s$ [**GS**], p.370, hence, *most* exotic $\mathbb{R}^4 s$ are not real algebraic in $\mathbb{R}^5$ or the interior of a smooth compact manifold. Exotic $\mathbb{R}^4 s$ are problematic at infinity: all *known* handle decompositions of exotic $\mathbb{R}^4 s$ are infinite [**GS**], p.366, all *known* exotic $\mathbb{R}^4 s$ contain a compact subset that cannot be contained in the bounded region formed by any smoothly embedded S^3 ("Property ★" [**MV**]), and *every* exotic $\mathbb{R}^4$ contains a compact subset that cannot be contained in any smoothly embedded D^4 (a "weak Property ★" [**M1**], p.168, see also [**GS**], p.366). It is unknown whether every exotic $\mathbb{R}^4$ possesses Property ★.

3. Algebraic Regular Neighborhoods

DEFINITION 1. *Suppose $X \subset Y$ are real algebraic sets with X compact. An algebraic regular neighborhood of X in Y is obtained as follows. Pick any proper rational function $p : Y \to \mathbb{R}$ with $X = p^{-1}(0)$ then for small enough $\epsilon > 0$, the set $p^{-1}([-\epsilon, \epsilon])$ is an algebraic regular neighborhood of X in Y.*

Algebraic regular neighborhoods are explored in [**D**]. They are unique up to isotopy. They are mapping cylinder neighborhoods. In our context, X will always contain the singular points of Y so $Y - X$ is a smooth manifold. Then ϵ small enough just means that p has no critical values in $[-\epsilon, 0) \cup (0, \epsilon]$, which easily implies independence of ϵ up to isotopy.

A related notion is an algebraic regular neighborhood of infinity for a real algebraic set Y. This is $p^{-1}((-\infty, -R] \cup [R, \infty))$ for a proper rational function p and large enough R. Algebraic regular neighborhoods of infinity are unique and collar the ends of Y, since they are algebraic regular neighborhoods of points added when compactifying Y. An example of such a neighborhood is the intersection of Y with the complement of a sufficiently large open ball.

4. Constructing the ends of W

NOTATION 1. *Let x, (x, t), and (x, t, s) denote typical elements of $\mathbb{R}^n$, $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, and $\mathbb{R}^{n+2} = \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ respectively. Let $B^m_r(x)$ denote the closed ball of radius r centered at x in $\mathbb{R}^m$ and B^m_r denote $B^m_r(0)$. We let S^{n-1}_r denote the sphere of radius r about the origin in $\mathbb{R}^n$.*

DEFINITION 2. *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a polynomial. The homogenization of h is the polynomial $h^* : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ given by $h^*(x, t) = t^{\deg(h)} h\left(\frac{x}{t}\right)$. The polynomial h is said to be overt provided $h^*(x, 0) \neq 0$ for $x \neq 0$.*

Essential to the proof of Theorem 1 is the following lemma, the proof of which will take up the rest of this section. One way of looking at it is that the pair $(\mathbb{R}^{n+1}, Z)$ has its algebraic regular neighborhood of infinity diffeomorphic to $(S^n, \Sigma) \times (R, \infty)$.

LEMMA 2. *Let Σ^{n-1} be a closed smooth codimension one submanifold of S^n. Then there is an algebraic subset $Z \subset \mathbb{R}^{n+1}$ such that for all sufficiently large R, the pair $(S^n_R, S^n_R \cap Z)$ is diffeomorphic to (S^n, Σ^{n-1}). In fact there is a proper imbedding $h\colon S^n \times [R, \infty) \to \mathbb{R}^{n+1}$ so that $h^{-1}(Z) = \Sigma^{n-1} \times [R, \infty)$ and $|h(x,t)| = t$ for all (x,t).*

REMARK 3. *The orientable, codimension 2 version of Lemma 2 already follows from known results: for an orientable 'knot' $M^n \hookrightarrow S^{n+2}$, a generalized Seifert surface (i.e. a compact, connected, smooth manifold $N^{n+1} \hookrightarrow S^{n+2}$ with trivial normal bundle such that $\partial N^{n+1} = M^n$) always exists (see, for instance, the Introduction and Section 27 of* [**R**]*), coupling this with Theorem 0.2 of* [**AK1**] *and inversion through the sphere, the result follows. In codimension ≥ 3, the problem is open (the interested reader may refer to* [**AK2**]*, for example).*

The outline of the proof of Lemma 2 is as follows. First, by deleting a point from S^n we will think of Σ^{n-1} as being a submanifold of $\mathbb{R}^n$. The major effort is then to construct a real algebraic set $V \subset \mathbb{R}^n \times [0, \infty)$ so that for $t > 0$ small enough, $V \cap \mathbb{R}^n \times t$ is isotopic to $\Sigma^{n-1} \times t$. Now put V in projective space $\mathbb{RP}^{n+1}$ then delete the subspace $t = 0$ to obtain an affine algebraic set Z. Now $Z \cap \mathbb{R}^n \times t$ is isotopic to $\Sigma^{n-1} \times t$ for all sufficiently large $t > 0$. By uniqueness of algebraic regular neighborhoods at infinity we then see that $(S^n_R, Z \cap S^n_R)$ is diffeomorphic to (S^n, Σ^{n-1}) for large enough radii R. Now for the details.

Let $Y^n \subset \mathbb{R}^n$ be the compact codimension zero submanifold so that $\partial Y^n = \Sigma^{n-1}$. Following Lemma 3.2 of [**AK1**] (and its proof), there is a finite collection of smoothly embedded disks D^n_α, $\alpha \in A$, in $int(Y^n)$ such that:

- The boundaries $S^{n-1}_\alpha = \partial D^n_\alpha$, $\alpha \in A$, are in general position and
- Y^n minus some other finite disjoint collection of disks is a smooth regularneighborhood of $\cup_\alpha S^{n-1}_\alpha$ in Y^n.

Essentially, the disks D^n_α, $\alpha \in A$ are obtained from a smooth triangulation of Y^n by choosing one disk D^n_α to engulf each simplex not touching $\Sigma^{n-1} = \partial Y^n$.

We define a finite abstract graph $\mathcal{G}$. The vertices v_j, $j \in B \subset \mathbb{Z}^+$, of $\mathcal{G}$ are the connected components of $Y^n - \cup_\alpha S^n_\alpha$. We order them so for $j \leq m$, v_j is a connected component of $Y^n - \cup_\alpha D^n_\alpha$, and for $j > m$, v_j is a connected component of $\cup_\alpha D^n_\alpha - \cup_\alpha S^n_\alpha$. We put an edge between two vertices v_i and v_j if and only if the intersection of the closures of their corresponding components contains an $n-1$ dimensional subset (that is, their corresponding components are nontrivially adjacent). The crucial property here is that we could actually realize the graph as a subset of Y^n, the vertices as points in their component and the edges as smooth curves between these points which are transverse to $\cup_\alpha S^n_\alpha$ and intersect it in a single point. We can take a subgraph $\mathcal{F} \subset \mathcal{G}$, the disjoint union of m trees $\mathcal{F}_i$ so that $v_i \in \mathcal{F}_i$ for $i \leq m$, and so $\mathcal{F}$ contains all the vertices of $\mathcal{G}$. Deleting a smooth regular neighborhood of $\mathcal{F}$ from Y gives us a manifold isotopic to Y as long as we chose $v_i \in \partial Y^n$ for $i \leq m$. See Figure 1.

ASSERTION 1. *After a small isotopy, we can assume $\cup_\alpha S^{n-1}_\alpha = p^{-1}(0)$ where $p : \mathbb{R}^n \to \mathbb{R}$ is an overt polynomial.*

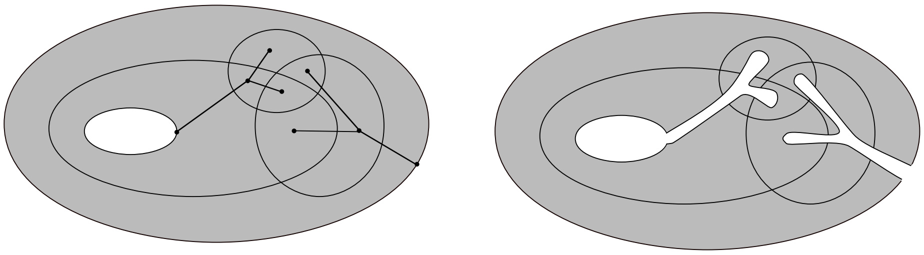

FIGURE 1. Y^n with $\mathcal{F}$ and with a regular neighborhood of $\mathcal{F}$ deleted

PROOF. By Theorem 2.8.2 of [**AK3**] we may suppose that each S_α^{n-1} is a nonsingular real algebraic set, hence it is $p_\alpha^{-1}(0)$ for some polynomial $p_\alpha\colon \mathbb{R}^n \to \mathbb{R}$ so that $\nabla p_\alpha \neq 0$ on S_α^{n-1}. We may suppose $p_\alpha > 0$ outside B^n. This polynomial p_α may not be overt, but if it is not we may replace it by $p_\alpha(x) + \epsilon|x|^{2k}$ for small $\epsilon > 0$ and $2k >$ degree of p_α and it will be overt. Now just let p be the product of all the p_α. □

Let $\mathcal{E}$ be the set of edges of $\mathcal{F}$ and let $|\mathcal{E}|$ be the number of edges in $\mathcal{E}$. If $\deg p \leq |\mathcal{E}|$ replace $p(x)$ with $(1+|x|^2)^k p(x)$ for a large enough k so that $\deg p > |\mathcal{E}|$. For each edge $e \in \mathcal{E}$, let x_e be the point of intersection of the edge with $\cup_\alpha S_\alpha^n$. Let $r : \mathbb{R}^n \to \mathbb{R}$ be the polynomial of degree $2\,|\mathcal{E}|$ given by:

$$r\,(x) = \prod_{e\in\mathcal{E}} |x - x_e|^2\,.$$

ASSERTION 2. *We may choose analytic coordinates in a neighborhood U_e of each x_e so that in these coordinates, $r(x) = |x|^2$ and $p(x) = \alpha_e(x_n)$ for some diffeomorphism $\alpha_e\colon (\mathbb{R},0) \to (\mathbb{R},0)$.*

PROOF. By induction there are a $k \leq n$ and analytic coordinates so $r(x) = \sum_{i=1}^{k-1} x_i^2 + h(x_k, \cdots x_n)$, r(0)=0, and $p(x) = x_n$. Since the Hessian of r is positive definite, $\partial^2 h/\partial x_k^2 \neq 0$ so by the implicit function theorem there is a smooth function $\beta(x_{k+1}, \ldots, x_n)$ so that $\partial h/\partial x_k(\,\beta(x_{k+1}, \ldots, x_n),\, x_{k+1}, \ldots, x_n) = 0$. Replace the coordinate x_k by the new coordinate $u = x_k - \beta$. Then $\partial h/\partial u = \partial h/\partial x_k$ vanishes on $u = 0$, so $h = u^2 h_1(u, x_{k+1}, \ldots, x_n) + h_2(x_{k+1}, \ldots, x_n)$ by Taylor's theorem. Now replace the coordinate u with the coordinate $v = u\sqrt{h_1}$ and the induction step is complete. Note that the coordinate x_n remains unchanged until the very last induction step. In this step $u = x_n$ and we let the germ of α_e be the inverse of the map $x_n \mapsto x_n\sqrt{h_1(x_n)}$. □

ASSERTION 3. *Let $g(x,t) = p^2(x) + bt^2 - 2ctr(x)$ with positive constants b and c to be determined below. Let $V = g^{-1}(0)$. Then:*

- *$V \cap \mathbb{R}^n \times (0,1] \subset Nonsing V$.*
- *The pair $(\mathbb{R}^n \times (0,1], V \cap \mathbb{R}^n \times (0,1])$ is diffeomorphic to $(\mathbb{R}^n, \Sigma^{n-1}) \times (0,1]$.*
- *$V \subset \mathbb{R}^n \times [0,\infty)$.*

PROOF. Before giving a careful, but boring and opaque proof, we'll give a rough idea why this works. For each t let $V_t = V \cap \mathbb{R}^n \times t$ and let N_t be the set of x so $g(x,t) \leq 0$, then N_t is compact and $V_t = \partial N_t$. Now N_t satisfies the equation $p^2(x) \leq \beta(x)$ where $\beta(x) = 2ctr(x) - bt^2$. The constants b and c will be small so β is small. If we are in a region where $\beta > 0$ then locally N_t is given roughly

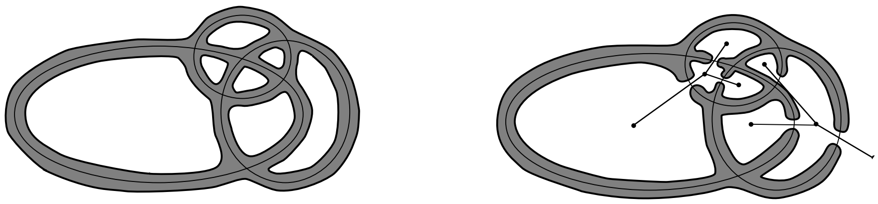

FIGURE 2. A regular neighborhood of $p^{-1}(0)$ and N_t

by $-d \leq p(x) \leq d$ for some small d, so N_t looks like the regular neighborhood $p^{-1}([-d,d])$ of $p^{-1}(0)$. But $\beta \leq 0$ only where r is small. There we may use the local coordinates given in Assertion 2. In these coordinates, V_t is roughly a hyperboloid $\Sigma_{i=1}^{n-1} x_i^2 - a x_n^2 = a'$. This has the effect of boring a hole through a regular neighborhood of $p^{-1}(0)$, in other words deleting a regular neighborhood of an arc going from one edge of the regular neighborhood to the other. So in the end, N_t is obtained from a regular neighborhood of $p^{-1}(0)$ by deleting a regular neighborhood of each edge in $\mathcal{E}$. But a regular neighborhood of $p^{-1}(0)$ is obtained from Y^n by deleting a disc around each vertex v_i with $i > m$. Thus N_t is obtained from Y^n by deleting a regular neighborhood of $\mathcal{F}$, and so N_t is isotopic to Y^n. Consequently, V_t is isotopic to $\Sigma^{n-1} = \partial Y^n$. See Figure 2.

Now for the details. Pick $\epsilon > 0$ so that $r^{-1}([0,2\epsilon]) \subset \bigcup_{e\in\mathcal{E}} U_e$. Since p is overt, we know it is proper. Let R be the maximum of $|\nabla r|$ on the compact set $p^{-1}([-1,1])$. Note that $|\nabla p(x)|/|p(x)| \to \infty$ as $p(x) \to 0$. This is because near a point of $p^{-1}(0)$ there are local coordinates so $p(x) = \prod_{i=k}^n x_i$ and in these coordinates we have $|\nabla p(x)|/|p(x)| = \sqrt{\sum_{i=k}^n 1/x_i^2}$. Consequently we may choose a $\delta \in (0,1)$ so that $|\nabla p(x)|/|p(x)| > R/\epsilon$ whenever $|p(x)| \leq \delta$.

Now since p and r are overt and $2\deg p > \deg r$ we know $p^2(x)/r(x) \to \infty$ as $x \to \infty$. Consequently we may choose $c \in (0,1)$ so that so that $2c < p^2(x)/r(x)$ whenever $|p(x)| \geq \delta$. We also require that $c < \delta^2/\epsilon$ and $\sqrt{2c} < \gamma_e(t)$ for all $e \in \mathcal{E}$ if $t^2 \leq \epsilon$, where $\gamma_e(t) = \alpha_e(t)/t$. Now let $b = c\epsilon$.

The first step is to show $V \cap \mathbb{R}^n \times (0,1] \subset \mathrm{Nonsing}V$ and the coordinate t as a function on $V \cap \mathbb{R}^n \times (0,1]$ has no critical points. Consequently there is an isotopy $h_t \colon \mathbb{R}^n \to \mathbb{R}^n$, $t \in (0,1]$ with compact support so that h_1 is the identity and $h_t(V_1) = V_t$. (You can get h_t by integrating a vector field $(v,-1)$ on $\mathbb{R}^n \times (0,1]$ which is tangent to V.) It suffices to show that whenever $g(x,t) = 0$ and $0 < t \leq 1$ then $\nabla_x g(x,t) \neq 0$. Here ∇_x denotes the gradient in the x variables. Note $g(x,t) = 0$ implies $p^2(x)/r(x) < 2ct$ so $p^2(x) < \delta^2$ by our choice of c. So suppose $\nabla_x g(x,t) = 0$. Then:

$$0 = \nabla_x g(x,t) = 2p\nabla p - 2ct\nabla r$$

so:

$$R/\epsilon < |\nabla p|/|p| = ct|\nabla r|/p^2 \leq ctR/p^2(x)$$

so we have $p^2(x) < ct\epsilon$. But then:

$$r(x) = (p^2(x) + bt^2)/(2ct) < \epsilon/2 + bt/(2c) = \epsilon(1+t)/2 \leq \epsilon.$$

So x must be in some U_e. In local coordinates we then have:

$$0 = \nabla_x g(x,t) = (-4ctx_1, \ldots, -4ctx_{n-1}, -4ctx_n + 2\alpha_e(x_n)\alpha_e'(x_n))$$

from which we see that $x_i = 0$ for $i < n$. But also:

$$0 = g(x,t) = \alpha_e(x_n)^2 + bt^2 - 2ctx_n^2.$$

So $2ct \geq \gamma_e^2$, contradicting our choice of c. So $\nabla_x g \neq 0$ on $V \cap \mathbb{R}^n \times (0,1]$ as required.

So it only remains to show that V_1 is isotopic to Σ^{n-1}. Let $V_1^+ = V_1 \cap \{x \mid p^2(x) \geq b\}$ and $p^{-2}(b) = p^{-1}(\{-\sqrt{b}, \sqrt{b}\})$. What we will show is that V_1^+ is diffeomorphic to $p^{-2}(b)$ with two discs removed for every $e \in \mathcal{E}$. Moreover V_1 is obtained from V_1^+ by gluing a one handle between each pair of these discs. But $p^{-2}(b)$ is the boundary of a regular neighborhood of $p^{-1}(0)$, which is Σ disjoint union a collection of spheres. Just as in [**AK3**] the one handles have the effect of connected summing these boundary components and we end up with V_1 being a manifold isotopic to Σ.

For each $e \in \mathcal{E}$ and $k = \pm 1$, let $D_{ke} = U_e \cap p^{-1}(k\sqrt{b}) \cap r^{-1}([0,\epsilon])$ which in the local coordinates around x_e is:

$$D_{ke} = \{x \mid x_n = b_k \text{ and } \sum_{i=1}^{n-1} x_i^2 \leq \epsilon - b_k^2\}$$

where $b_k = \alpha_e^{-1}(k\sqrt{b})$. Note $\alpha_e(\pm\sqrt{\epsilon})^2 = \epsilon\gamma_e(\pm\sqrt{\epsilon})^2 > 2c\epsilon = 2b$, so $|b_k| < \sqrt{\epsilon}$ and so each D_{ke} is an $n-1$ disc. Now let $E_e = U_e \cap V_1 \cap p^{-1}([-\sqrt{b}, \sqrt{b}])$ which in the local coordinates around x_e is:

$$E_e = \{x \mid b_{-1} \leq x_n \leq b_1 \text{ and } \sum_{i=1}^{n-1} x_i^2 = \epsilon/2 + x_n^2(\gamma_e^2(x_n)/(2c) - 1)\}.$$

Recall $\gamma_e^2 > 2c$ so each E_e is a one handle $[-1,1] \times S^{n-2}$ attached to $\partial D_{1e} \cup \partial D_{-1e}$. We claim that V_1^+ is isotopic to $p^{-2}(b) \cap r^{-1}([\epsilon,\infty))$ rel $p^{-2}(b) \cap r^{-1}(\epsilon) = \bigcup_{e \in \mathcal{E}} \partial D_{1e} \cup \partial D_{-1e}$. So once we show this, then we know V_1 is isotopic to $p^{-2}(b)$ with a one handle attached near each x_e. But this is isotopic to Σ.

The isotopy from V_1^+ to $p^{-2}(b) \cap r^{-1}([\epsilon,\infty))$ is obtained by integrating the vector field $-p\nabla p$, which points into the region $\{x \mid p^2(x) \geq b \text{ and } g(x,1) \leq 0\}$ on V_1^+ and out on $p^{-2}(b) \cap r^{-1}([\epsilon,\infty))$. To see it points in on V_1^+, recall that we saw above that $|p(x)| < \delta$ if $g(x,1) = 0$. But this means $|\nabla p(x)|/|p(x)| > R/\epsilon$ by our choice of δ so:

$$\begin{aligned} -p\nabla p \cdot \nabla_x g &= -p^2|\nabla p|^2 + cp\nabla r \cdot \nabla p \\ &\leq -2cr|\nabla p|^2 + c|p||\nabla r||\nabla p| \leq -c|p\nabla p|(2\epsilon|\nabla p|/|p| - R) \\ &< -cR|p\nabla p| < 0. \end{aligned}$$

□

There are a number of routes to obtaining the desired Z from V. One route is to use Proposition 2.6.1 of [**AK3**] to algebraically crush V_0 to a point, then invert through the sphere to send this point to infinity. This would correspond to the transformation $(x,t) \mapsto (x,1)/(t + t|x|^2)$. We'll take another route, corresponding to the transformation $\theta(x,t) = (x/t, 1/t)$.

Let $g^*(x,t,s)$ be the homogenization of g. Let $G(x,t) = g^*(x,1,t)$ and let $Z = G^{-1}(0)$. Note that $Z - \mathbb{R}^n \times 0 = \theta(V - V_0)$.

We want to show for large enough radii R that $(S_R^n, S_R^n \cap Z)$ is diffeomorphic to (S^n, Σ^{n-1}). But this follows from uniqueness of algebraic regular neighborhoods of infinity. Since [**D**] does not explicitly deal with regular neighborhoods of pairs we will outline the argument which is similar to arguments in [**D**]. Consider $D = \{(x,t) \mid 1 \le t \text{ and } |x|^2 + t^2 \le R^2\}$. The boundary of D is $D_+ \cup D_-$ where D_- is the disc $\{(x,1) \mid R^2 - 1 \ge |x|^2\}$ and D_+ is the spherical cap $\{(x,t) \in S^n_R \mid 1 \le t\}$. For large enough R there is a vector field $(w,1)$ on D which is tangent to Z and points outward on D_+ and inward on D_-. Integrating this vector field gives a diffeomorphism between the pairs $(D_-, D_- \cap Z)$ and $(D_+, D_+ \cap Z)$. Note that $D_- \cap Z = V_1 \approx \Sigma$ and $D_+ \cap Z = S^n_R \cap Z$ and consequently $(S^n_R, S^n_R \cap Z)$ is diffeomorphic to (S^n, Σ^{n-1}).

This completes the proof of Lemma 2.

5. Proof of Theorem 1

First, suppose $int\,(X^n)$ is diffeomorphic to a nonsingular, real algebraic subset of $\mathbb{R}^{n+1}$. Then for large r, $\partial B^{n+1}_r \pitchfork int\,(X^n)$ in a smooth manifold Σ^{n-1} that collars $int\,(X^n)$ at infinity (see section 2 of [**M3**] and use stereographic projection to compactify $\mathbb{R}^n$). Let $X^n_0 = B^{n+1}_r \cap int\,(X^n)$. Now, ∂X^n and ∂X^n_0 are not necessarily diffeomorphic, however, it is not difficult to see that they are invertibly cobordant (see [**St**]), say by $(W; \partial X^n, \partial X^n_0)$. By definition, $(W; \partial X^n, \partial X^n_0)$ embeds smoothly in $\partial X^n_0 \times [0,1]$. Using this and the fact that there is a smooth, proper embedding $X^n_0 \hookrightarrow B^{n+1}_r \approx D^{n+1}$, it follows that there is $X^n \hookrightarrow D^{n+1}$ a smooth, proper embedding, as desired.

The other direction follows from Lemma 2 and the following:

Lemma 3. *Let $V \subset \mathbb{R}^n$ be a codimension one real algebraic set with SingV compact. Let $M \subset \mathbb{R}^n$ be a proper smooth codimension one submanifold so that for some R, $M - B^n_R = V - B^n_R$. Then there is a nonsingular real algebraic set $W \subset \mathbb{R}^n$ properly isotopic to M. In fact, we may suppose there is a smooth isotopy $h_t \colon \mathbb{R}^n \to \mathbb{R}^n$ and a radius R' so that h_0 is the identity, $h_1(M) = W$, and $|h_t(x)| = |x|$ whenever $|x| \ge R'$.*

Proof. Pick a polynomial $p \colon \mathbb{R}^n \to \mathbb{R}$ generating the ideal of polynomials vanishing on V. So $p^{-1}(0) = V$ and the only solutions to $p = 0$ and $\nabla p = 0$ are Sing$V$. Let $r(x) = |x|^2$. Let $q(x) = p^2(x) + |\nabla p|^2|x|^2 - (\nabla p \cdot x)^2$. Then $q^{-1}(0)$ is the set of points in V where ∇p and x are linearly dependent, so it is the union of Sing$V$ and the critical points of $r|_{\text{Nonsing}V}$. Thus $q^{-1}(0)$ is compact which means by Lemma 2.1.5 of [**AK3**] that for some radius R'' and integer $m \ge 0$, $q(x) \ge 3|x|^{-2m}$ whenever $|x| \ge R''$. Since M separates $\mathbb{R}^n$ we may find a smooth function $f \colon \mathbb{R}^n \to \mathbb{R}$ and a radius $R' > R''$ so that 0 is a regular value of f, $f^{-1}(0) = M$, and $f(x) = p(x)$ if $|x| \ge R'$. Pick an integer $k > 1 + m + \text{degree}(p)$. Choose $\epsilon > 0$ so $|\nabla f(x)| > \epsilon$ whenever $|f(x)| < \epsilon$ and $|x| \le R'$. Also make sure that $|\nabla p(x)| < |x|^{2k-2m-2}/\epsilon$ whenever $|x| \ge R'$. Also make sure that $\epsilon < (R')^{2k-m}$. By Lemma 2.8.1 of [**AK3**] applied to $f - p$ there is an entire rational function $u \colon \mathbb{R}^n \to \mathbb{R}$ approximating f so $|f(x) - u(x)| < \epsilon(1+|x|^2)^{-k}$ and $|\nabla f(x) - \nabla u(x)| < \epsilon(1+|x|^2)^{-k}$ for all $x \in \mathbb{R}^n$. Let $W = u^{-1}(0)$.

Let $F(x,t) = tu(x) + (1-t)f(x)$. We claim there is a vector field $(v(x,t),1)$ on $\mathbb{R}^n \times [0,1]$ tangent to $F^{-1}(0)$ so that $v(x,t) \cdot x = 0$ if $|x| \ge R'$. Then integrating this vector field gives the isotopy h_t.

It suffices to construct v locally. Locally we may take $v = 0$ if $F \ne 0$. If $F(x,t) = 0$ and $|x| < R'$ we will locally take $v(x,t) = \alpha(x,t)\nabla f$ for an appropriate

α, in particular:

$$\alpha(x,t) = (f(x) - u(x))/(|\nabla f|^2 - t\nabla f \cdot (\nabla f - \nabla u)).$$

If $F(x,t) = 0$ and $|x| \geq R'$ we will locally take $v(x,t) = \alpha(x,t)v'(x,t)$ where $v'(x,t) = |x|^2\nabla f - (\nabla f \cdot x)x$ for an appropriate α, in particular:

$$\alpha(x,t) = (f(x) - u(x))/v'(x,t) \cdot ((1-t)\nabla f - t\nabla u).$$

Note $p^2(x) = f^2(x) = t^2(f(x) - u(x))^2 \leq \epsilon^2|x|^{-4k} < |x|^{-2m}$ so the denominator is nonzero since:

$$\begin{aligned} v'(x,t) \cdot ((1-t)\nabla f - t\nabla u) &= q(x) - p^2(x) + tv'(x,t) \cdot (\nabla f - \nabla u) \\ &> 3|x|^{-2m} - |x|^{-2m} - 2|x|^2|\nabla p||\nabla f - \nabla u| \\ &> 2|x|^{-2m} - 2\epsilon|x|^{2-2k}|\nabla p| > 0. \end{aligned}$$

□

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