Constructing Strange Real Algebraic Sets

by S. Akbulut* and H. King**

Recall that algebraic sets are characterized by resolution tower structures [**AK1**], [**AK2**]. A topological space with such a structure is called a **TR** space for short. In low dimensions existence of resolution tower structures can often be detected by combinatorial invariants of the underlying topological spaces. For example, let X be a locally cone-like Euler stratified space (i.e., a locally conelike stratified set so that the Euler characteristics of the links of all strata are even). Then for every component of the codimension one stratum U of X we define $\kappa_X(U)$ to be the number of points in the link of U. For every component of the codimension two stratum W of X we define $\alpha_i(W)$ to be the number of vertices v of the link L of W so that $\kappa_L(v) = i \bmod 8$, where $i = 0, \ldots, 7$. Then define

$$\begin{aligned}
\epsilon_0(W) &= \alpha_0(W) \bmod 2 \\
\epsilon_1(W) &= \alpha_6(W) \bmod 2 \\
\epsilon_2(W) &= (\alpha_0(W) + \alpha_4(W))/2 \bmod 2 \\
\epsilon_3(W) &= (\alpha_2(W) + \alpha_6(W))/2 \bmod 2
\end{aligned}$$

and define $\epsilon_X(W) = (\epsilon_0(W), \epsilon_1(W), \epsilon_2(W), \epsilon_3(W)) \in \mathbf{Z}_2^4$. It is an easy exercise to show that $\epsilon_X(W)$ is well defined, i.e., $\alpha_0 + \alpha_4$, and $\alpha_2 + \alpha_6$ are even. For every component V of the codimension three stratum of X and every $\epsilon \in \mathbf{Z}_2^4$ we define $\beta_\epsilon(V) \in \mathbf{Z}_2^4$ to be the number of vertices v of the link L of V with $\epsilon_L(v) = \epsilon$.

Theorem. *([**AK2**]) For any compact topological space X, the following are equivalent:*

a) X is homeomorphic to a real algebraic set of dimension ≤ 3.

*b) X is a **TR** space of dimension ≤ 3.*

c) X is a locally cone-like Euler stratified space of dimension ≤ 3 and for all codimension three strata x of X we have:

$$\begin{aligned}
\beta_{1110}(x) + \beta_{1111}(x) &= 0 \\
\beta_{0100}(x) + \beta_{0101}(x) &= 0 \\
\beta_{1000}(x) + \beta_{1001}(x) &= 0 \\
\beta_{1100}(x) + \beta_{1101}(x) &= 0
\end{aligned}$$

Hence whether or not a stratified space of dimension less then four is homeomorphic to an algebraic set can be decided by purely combinatorial data. In [**AK1**], [**AK2**] a

* Supported in part by the N.S.F.

** Supported in part by the Graduate Research Board, University of Maryland

similar characterization, with homeomorphism replaced by a stratified set isomorphism, is also given. (One needs all β_ϵ's to be 0 and if all strata have trivial normal bundle then this suffices.)

This gives some surprising examples of real algebraic sets. For example, in **[CK]** this was used to give an example of a real algebraic set which is topologically imbedded in $\mathbf{R}^4$ but cannot be an algebraic subset of $\mathbf{R}^4$. The purpose of this paper is to indicate a construction of such algebraic sets which, for these special cases, is simpler than the general construction given in **[AK2]**. It falls short of giving explicit equations, but comes much closer to an explicit description.

Let us now proceed with the construction, which follows the spirit, but differs quite a bit from the letter, of that in **[AK2]**. We shall construct below a compact three dimensional manifold M with boundary and closed subsets A, B_0 and B_1 so that:

1) A is a spine of M, in fact there is a diffeomorphism $h: \partial M \times (0,1] \to M - A$ so that $h(x,1) = x$ for all $x \in \partial M$ and $h^{-1}(B_i) = (\partial M \cap B_i) \times (0,1]$.
2) B_0 and B_1 are disjoint.
3) If you take ∂M, crush $\partial M \cap B_0$ to a point and then crush $\partial M \cap B_1$ to a point, you obtain a two dimensional sphere.
4) Everything can be made algebraic. In particular, there is a compact nonsingular three dimensional real algebraic set $V \subset \mathbf{R}^n$, algebraic subsets C, D_0 and D_1 and an imbedding $g: M \to V$ so that $C = g(A)$, $B_i = g^{-1}(D_i)$ and $D_0 \cap D_1 = \emptyset$.

By **[AK2]** we may as well assume that the V in 4) is projectively closed, since any compact real algebraic set is isomorphic to a projectively closed algebraic set. Alternatively, using the fact we see below that A and B_i are unions of codimension one submanifolds in general position, this is a consequence of Theorem 2.10 of **[AK4]**. So there is an overt polynomial $r: \mathbf{R}^n \to \mathbf{R}$ with $V = r^{-1}(0)$. Overtness means that if r has degree d and r_d is the sum of the monomials of r of degree d, then $r_d^{-1}(0) = \{0\}$. Pick polynomials $q_i: \mathbf{R}^n \to \mathbf{R}$ so that $q_i(x) \geq 0$ for all x and so $q_i^{-1}(0) = C \cup D_i$. For example, q_i could be the sum of squares of generators of the ideal of polynomials vanishing on $C \cup D_i$. Let $q = q_0 + q_1$ and pick an integer e bigger than the degrees of q_0 and q_1. Note that $C = q^{-1}(0)$.

Let $\alpha: V \to \mathbf{R}^{n+2}$ be the map

$$\alpha(x) = (\, q_0(x)q_1(x)x \,,\, q_0(x) \,,\, q_1(x) \,)$$

and let $Z \subset \mathbf{R}^{n+2}$ be the algebraic set

$$Z = \{\, (y, t_0, t_1) \mid y = 0 \text{ and } t_0 t_1 = 0 \,\}.$$

We claim that $\alpha(V) \cup Z$ is a real algebraic set and near 0 it has strange behavior which was remarked upon in **[CK]**, see their question (¿_d). In particular, the line segment

$\{y = 0, t_0 = 0, t_1 < 0\}$ is open in $\alpha(V) \cup Z$ but $\{y = 0, t_0 = 0, t_1 > 0\}$ is contained in a part of $\alpha(V) \cup Z$ which is a topological manifold. See Figure 1, which is a fairly truthful representation. In particular, $\alpha(V)$ lies over the first quadrant and $Z - \alpha(V)$ consists of two half lines.

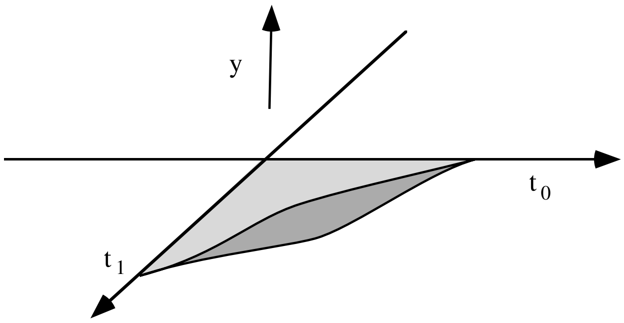

Figure 1: $\alpha(V) \cup Z$

To see our claim that $\alpha(V) \cup Z$ is a real algebraic set, let $r' \colon \mathbf{R}^{n+2} \to \mathbf{R}$ and $q'_i \colon \mathbf{R}^{n+2} \to \mathbf{R}$ be the polynomials

$$r'(y, t_0, t_1) = (t_0 t_1)^d \, r(y/t_0 t_1)$$

$$q'_i(y, t_0, t_1) = (t_0 t_1)^e \, (t_i - q_i(y/t_0 t_1))$$

Let W be the algebraic set $W = r'^{-1}(0) \cap q_0'^{-1}(0) \cap q_1'^{-1}(0)$. Then certainly $\alpha(V) \subset W$ and $Z \subset W$. But in fact $\alpha(V) \cup Z = W$. To see this, pick any $(y, t_0, t_1) \in W$. If $t_0 t_1 = 0$ then $0 = r'(y, t_0, t_1) = r_d(y)$ so $y = 0$ by overtness of r and hence $(y, t_0, t_1) \in Z$. But if $t_0 t_1 \neq 0$ then setting $x = y/(t_0 t_1)$ we have $r(x) = 0$ so $x \in V$ and $(y, t_0, t_1) = \alpha(x)$.

Now let us investigate the topology of W near 0. We think of W as the quotient space of $V \cup Z$ via the map α. Suppose $\alpha(x) = \alpha(x')$ for some $x, x' \in V$. If $x \notin C \cup D_0 \cup D_1$ then $q_0(x) q_1(x) \neq 0$. But $q_i(x') = q_i(x)$ and $q_0(x) q_1(x) x = q_0(x') q_1(x') x' = q_0(x) q_1(x) x'$ so $x = x'$. If $x \in D_i - C$ then $q_i(x) = 0$ and $q_{1-i}(x) \neq 0$ so $q_i(x') = 0$ and $q_{1-i}(x') \neq 0$ so $x' \in D_i - C$ also and $q(x) = q(x')$. If $x \in C$ then $q_0(x) = q_1(x) = 0$ so $q_0(x') = q_1(x') = 0$ and $x' \in C$ also. So the map α crushes C to a point and maps D_i to the half line $y = 0$, $t_i = 0$, $t_{1-i} \geq 0$.

For small enough $\epsilon > 0$ we have a homeomorphism $f \colon q^{-1}(\epsilon) \times (0, \epsilon] \to q^{-1}((0, \epsilon])$ so that $qf(x, t) = t$ and $f^{-1}(D_i) = (D_i \cap q^{-1}(\epsilon)) \times (0, \epsilon]$. This follows from Thom's first isotopy lemma, although in fact C and D_i are unions of codimension one submanifolds in general position so if one wanted, one could take f to be a diffeomorphism. Anyway we have an induced homeomorphism $f' \colon \alpha(q^{-1}(\epsilon)) \times (0, \epsilon] \to \alpha(q^{-1}((0, \epsilon]))$ where $f'(\alpha(x), t) = \alpha(f(x, t))$. Thus near 0, $\alpha(V)$ is the cone on $\alpha(q^{-1}(\epsilon))$. Consequently near 0, W is the cone on the disjoint union of $\alpha(q^{-1}(\epsilon))$ and two points. But $\alpha(q^{-1}(\epsilon))$ is obtained from $q^{-1}(\epsilon)$ by crushing $q^{-1}(\epsilon) \cap D_0$ to one point and $q^{-1}(\epsilon) \cap D_1$ to another. But this space must be the two sphere. The reason is that the two product structures f and gh give an invertible cobordism from the triple $(q^{-1}(\epsilon), q^{-1}(\epsilon)\cap D_0, q^{-1}(\epsilon)\cap D_1)$ to $(\partial M, \partial M\cap B_0, \partial M\cap B_1)$. But because the dimensions are so small the cobordism is trivial. So there is a homeomorphism between the two triples.

This algebraic set W has the following curious property. Consider one of the lines in Z, say $y = 0$, $t_0 = 0$. Then if $t_1 < 0$ it is locally open in W. But if $t_1 > 0$, then W is a three dimensional topological manifold nearby. Thus it has the curious behavior remarked upon in **[CK]**.

One could modify this example a bit to obtain the specific space mentioned in **[CK]**. Let $W' \subset \mathbf{R}^{n+3}$ be the union of $W \times 0$ and the plane $\{ (y, t_0, t_1, t_2) \mid y = 0 \text{ and } t_0 = 0 \}$. Then near 0, W' is the cone on the disjoint union of a point and the wedge of a 2-sphere and a circle. To obtain a real algebraic set homeomorphic to the suspension of this, one can use the Lemma below.

Lemma. *Let X be a real algebraic set and pick $x_0 \in X$. It is well known that x_0 has a neighborhood in X homeomorphic to the cone on some compactum L. Then there is a real algebraic set Y homeomorphic to the suspension of L.*

Proof: We may suppose $X \subset \mathbf{R}^n$ and $x_0 = 0$. For small enough $\epsilon > 0$ we know that $X \cap B_\epsilon$ is homeomorphic to the cone on $X \cap \partial B_\epsilon$ where B_ϵ is the ball around 0 of radius ϵ and ∂B_ϵ is its boundary sphere. We then take

$$Y = \{ (x, t) \in X \times \mathbf{R} \mid t^2 + |x|^2 = \epsilon^2 \}.$$

Clearly Y is the suspension of $X \cap \partial B_\epsilon$. But L and $X \cap \partial B_\epsilon$ are invertibly cobordant (see **[K]**) so their products with $\mathbf{R}$ are homeomorphic and hence their suspensions are homeomorphic also. ∎

We promised above the construction of the manifold M and its subsets A and B_i. We start out with part of the boundary of M. Let E be a two dimensional disc with three handles attached as in Figure 2. For convenience we do not smooth out corners in the figures although in reality they are smooth. Let E_1, E_2 and E_3 be as shown. Note that the quotient space $E/(E_1 \cup E_2 \cup E_3)$ is a disc. For another description, note that E is obtained from a disc by blowing up three times and the E_i's are the strict transforms of the exceptional divisors.

Next we consider the manifold $F = E \times [0, 1]$. We show it in Figure 3, although it does not quite imbed in $\mathbf{R}^3$. In F we have an immersed surface $H = (E_1 \cup E_2 \cup E_3) \times [0, 1]$. The manifold F will be part of M and H will be part of B_0.

Our next step is to add a handle to F as in Figure 4. In particular, letting $J = [-1, 1]$, we pick disjoint imbeddings $\varphi_i\colon J \times J \to E \times 1$ for $i = -1, 1$ so that $\varphi_{-1}(0, 0)$ is the point $(E_1 \cap E_3) \times 1$ and $\varphi_1(0, 0)$ is the point $(E_2 \cap E_3) \times 1$, so that $\varphi_i^{-1}(E_3 \times 1) = 0 \times J$,

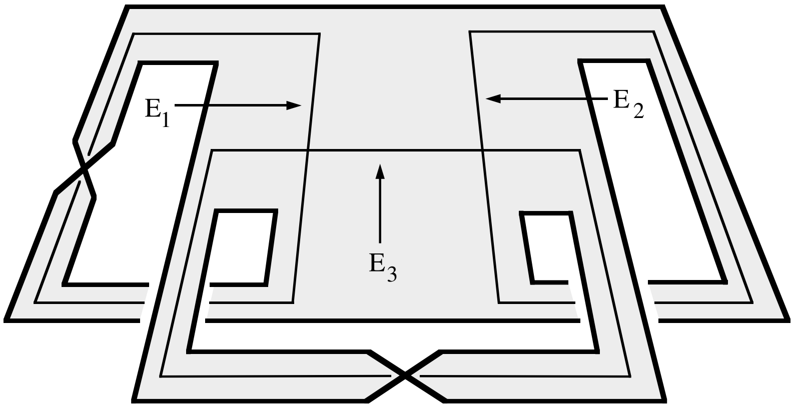

Figure 2: E

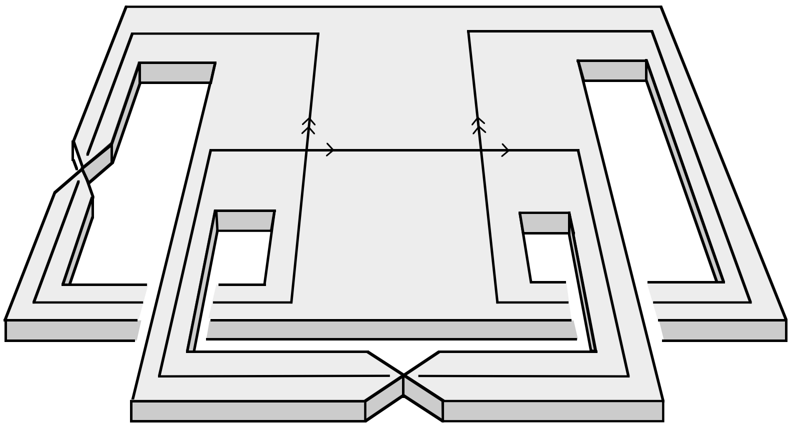

Figure 3: F

$\varphi_1^{-1}(E_2 \times 1) = J \times 0$ and $\varphi_{-1}^{-1}(E_1 \times 1) = J \times 0$. Furthermore the imbeddings are oriented as shown by the arrows in Figure 3. This gives the handle a twist if we try to imbed it in $\mathbf{R}^3$ as we do in Figure 4. We let $F' = F \cup J \times J \times J$ where we identify (i, s, t) with $\varphi_i(s, t)$ for $i = -1, 1$ and $(s, t) \in J \times J$. We let $H' = H \cup J \times 0 \times J \cup J \times J \times 0$ and $G' = 0 \times J \times J$. This G' will be part of A.

Note that the the upper part of the boundary of H', i.e., that part which is not in $E \times 0$, consists of two disjoint circles and each of these circles has nontrivial normal bundle in $\partial F'$ and intersects G' transversely in two points. Our next step is to attach a thickened cylinder to these two circles.

In particular, let K be a Moebius band with central circle K' and let K'' be two transverse arcs as in Figure 5. Then we can pick disjoint imbeddings $\theta_i \colon K \to \partial F'$ for $i = -1, 1$ so that $\theta_i^{-1}(H') = K'$ and $\theta_i^{-1}(G') = K''$. We now let $F'' = F' \cup J \times K$ where we identify (i, x) with $\theta_i(x)$ for $i = -1, 1$ and $x \in K$. We let $H'' = H' \cup J \times K''$ and $G'' = G' \cup J \times K'' \cup 0 \times K$.

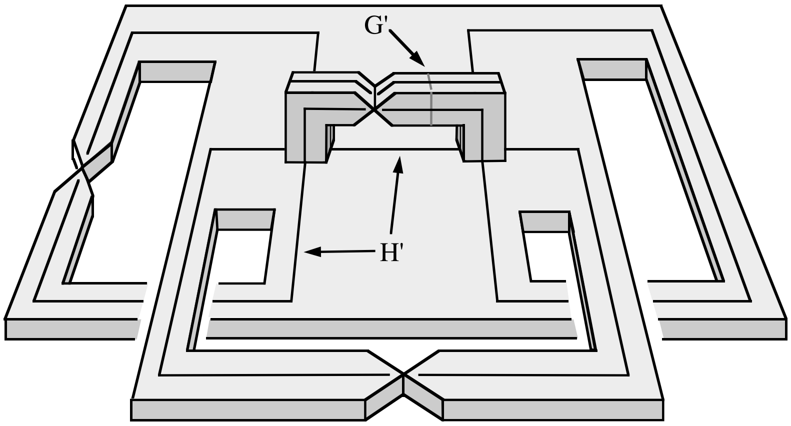

Figure 4: F'

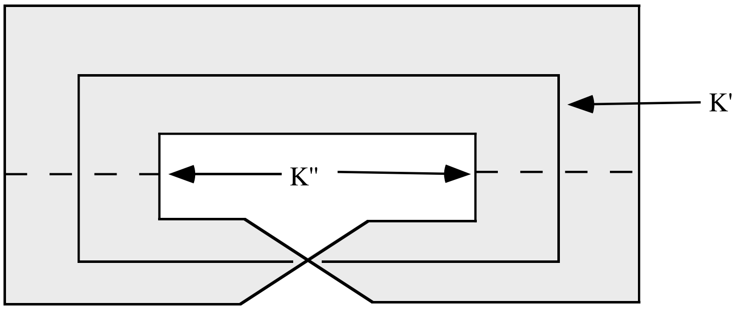

Figure 5: K

Our construction has the following property. There is a diffeomorphism $f\colon E\times(0,1]\to F''-G''$ so that $f(E\times 1)=E\times 0$ and $f((E_1\cup E_2\cup E_3)\times(0,1])=H''-G''$. For example, f can be obtained by integrating a vector field tangent to H'' and to $\partial F''-E\times 0$ and pointing inward on $E\times 0$.

We now let M be obtained by doubling F'' along $\partial F''_+=\mathrm{Cl}(\partial F''-E\times 0)$. In other words, $M=F''\times\{0,1\}\cup\partial F''_+\times[0,1]$. We let $A=G''\times\{0,1\}\cup\partial G\times[0,1]$ and $B_i=H''\times i$. Actually to make sure we can make B_i algebraic, by say [**AK4**], we must make sure B_i is a union of imbedded submanifolds which it is not. But it is easy to make it so by blowing up F'' with center the double points of H''.

Note that it is only necessary to make F'', G'' and H'' algebraic and then a simple construction allows us to make M, A and B_i algebraic. In particular, suppose we have a three dimensional real algebraic set X and algebraic subsets Y and Z of X and an imbedding $f\colon F''\to X$ so that $f^{-1}(Y)=G''$ and $f^{-1}(Z)=H''$. Suppose also that Z is projectively closed so that $Z=p^{-1}(0)$ for some overt polynomial p. Say p has degree d. Then for small $\epsilon>0$ let $V=\{\,(x,t)\in X\times\mathbf{R}\mid p(x)^2+t^{2d}=\epsilon^{2d}\,\}$ and $C=V\cap Y\times\mathbf{R}$, $D_0=Z\times(-\epsilon)$ and $D_1=Z\times\epsilon$. Then V is projectively closed and satisfies 4) above. The point is that $F''-H''$ is diffeomorphic to the product $\partial F''_+\times(0,1]$ via a diffeomorphism preserving G''. But for small $\epsilon > 0$, the set $p^{-1}((-\epsilon^d, 0) \cup (0, \epsilon^d))$ is also a product with $(0, 1]$. Consequently because the dimensions are so low that invertible cobordisms are trivial, we know $\partial F''_+$ imbeds in $p^{-1}(\pm\epsilon^d)$ in a way which takes $\partial G''$ to $p^{-1}(\pm\epsilon^d) \cap Y$. Consequently we see F'' doubled along $\partial F''_+$ in V. But this is just M.

A certain amount of staring at Figure 4 convinces one that F'' can be imbedded in $\mathbf{R}^4$. This is also seen from the fact that $G'' \cap H''$ is a spine of F'', i.e., $F'' - G'' \cap H'' = \partial F'' \times (0, 1]$. So F'' is just a thickening of $G'' \cap H''$ which is the curve in Figure 6.

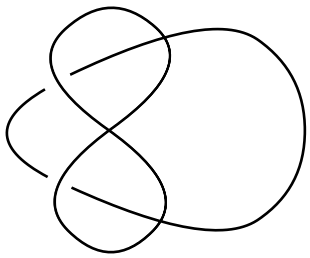

Figure 6: $G'' \cap H''$

Note that a variation on this example provides a real algebraic set homeomorphic to the suspension of the disjoint union of a point and the connected sum of three $\mathbf{RP}^2$'s. The only difference is that we replace α with the map

$$\alpha(x) = (\, q_0(x)x \,,\, q_0(x) \,,\, q_1(x) \,)$$

and replace Z by the algebraic set

$$Z = \{\, (y, t_0, t_1) \mid y = 0 \text{ and } t_0 = 0 \,\}.$$

[AK1] S. Akbulut and H. King, The topological classification of 3-dimensional algebraic sets, M.S.R.I. preprint 12912-85 (1985).

[AK2] S. Akbulut and H. King, The topology of real algebraic sets, M.S.R.I. book series (to appear).

[AK3] S. Akbulut and H. King, The topology of real algebraic sets, L'Enseignement Math., 29 (1983) 221-261.

[AK4] S. Akbulut and H. King, The topology of real algebraic sets with isolated singularities, Annals of math., Vol. 113 (1981), 425-446.

[CK] M. Coste and K. Kurdyka, On the link of a stratum in a real algebraic set, to appear in Topology.

[K] H. King, Real analytic germs and their varieties at isolated singularities, Inv. Math., Vol. 37 (1976), 193-199.

[S] D. Sullivan, Invariants of analytic spaces, Proc. Liverpool Singularities I, Lecture Notes 192, Springer-Verlag (1971) 165-168.