CONFIGURATION SPACES OF LINKAGES IN $\mathbb{R}^n$

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Abstract. This paper studies the configuration space of all possible positions of a linkage in $\mathbb{R}^n$. For example, it shows that for every compact algebraic set, there is a linkage whose configuration space is analytically isomorphic to a finite number of copies of the algebraic set. If flexible edges are allowed, any compact set given by polynomial equalities and inequalities is the configuration space of a linkage. This paper also studies semiconfiguration spaces of all possible positions of a finite number of points on a linkage. For example any compact semialgebraic set is such a semiconfiguration space.

1. Linkages

Loosely speaking, a linkage is an ideal mechanical device consisting of a bunch of stiff rods sometimes attached at their ends by rotating joints. A realization of a linkage in $\mathbb{R}^n$ is some way of placing this linkage in $\mathbb{R}^n$. The configuration space for a linkage is the space of all such realizations, which can be determined by looking at all possible positions of the ends of all the rods. A semiconfiguration space of a linkage is the space of all possible positions of only some of ends of the rods, we ignore the other ends. For example, what figure does a particular point on the linkage trace out?

In this paper we will give characterizations of configuration spaces and semiconfiguration spaces of linkages as well as of cabled linkages, for $n \geq 3$. (In a cabled linkage you also can attach flexible cables between rods.) The characterizations of these spaces for $n = 2$, planar linkages, was studied in [3], [1], and [2]. The results for $n \geq 3$ turn out to be analogous to the $n = 2$ results, but in a couple of places the proofs are different. In particular, we can completely characterize semiconfiguration spaces, we can characterize configuration spaces of cabled linkages up to analytic isomorphism, and we can characterize configuration spaces of linkages up to analytically trivial finite covers.

Let us now define linkages more precisely. Suppose L is a finite one dimensional simplicial complex, in other words, a finite set $\mathcal{V}(L)$ of vertices and a finite set $\mathcal{E}(L)$ of edges between certain pairs of vertices. An *abstract linkage* is a finite one dimensional simplicial complex L with a mapping $\ell\colon \mathcal{E}(L) \to (0,\infty)$. You should think of ℓ as giving the length of each edge. A *realization* of an abstract linkage (L,ℓ) in $\mathbb{R}^n$ is a mapping $\varphi\colon \mathcal{V}(L) \to \mathbb{R}^n$ so that $|\varphi(v) - \varphi(w)| = \ell(\overline{vw})$ for all edges $\overline{vw} \in \mathcal{E}(L)$.

We will often wish to fix some of the vertices of a linkage whenever we take a realization. So we say that a *classical linkage in* $\mathbb{R}^n$ is a foursome $\mathcal{L} = (L,\ell,V,\mu)$ where (L,ℓ) is an abstract linkage, $V \subset \mathcal{V}(L)$ is a subset of its vertices, and $\mu\colon V \to$

Date: November 23, 1998.
1991 *Mathematics Subject Classification.* Primary 14P05, 14P10; Secondary 57R99,14P20.

$\mathbb{R}^n$. So V is the set of fixed vertices and μ tells where to fix them. The configuration space of realizations is defined by:

$$\mathcal{C}(\mathcal{L}) = \left\{ \varphi\colon \mathcal{V}(L) \to \mathbb{R}^n \;\middle|\; \begin{array}{cl} \varphi(v) = \mu(v) & \text{if } v \in V \\ |\varphi(v) - \varphi(w)| = \ell(\overline{vw}) & \text{for all edges } \overline{vw} \in \mathcal{E}(\mathcal{L}) \end{array} \right\}$$

A *cabled linkage in* $\mathbb{R}^n$ is a quintuple (L, ℓ, V, μ, F) where (L, ℓ, V, μ) is a classical linkage and $F \subset \mathcal{E}(L)$. We will think of the edges in F as being flexible rather than rigid. A physical model for such a cabled linkage would have the edges in $\mathcal{E}(L) - F$ be rigid rods but the edges in F are just ropes or cables connecting two vertices. Thus in a realization, two vertices connected by an edge e in F would only be constrained to have distance $\leq \ell(e)$. The configuration space of a cabled linkage is given by:

$$\mathcal{C}(\mathcal{L}) = \left\{ \varphi\colon \mathcal{V}(L) \to \mathbb{R}^n \;\middle|\; \begin{array}{cl} \varphi(v) = \mu(v) & \text{if } v \in V, \\ |\varphi(v) - \varphi(w)| \leq \ell(\overline{vw}) & \text{if } \overline{vw} \in F \\ |\varphi(v) - \varphi(w)| = \ell(\overline{vw}) & \text{if } \overline{vw} \in \mathcal{E}(L) - F \end{array} \right\}$$

From now on, the word linkage will refer to a cabled linkage in $\mathbb{R}^n$. If we wish to refer to a linkage without any flexible edges, we will call it a classical linkage. If F is empty we get a classical linkage.

If $W \subset \mathcal{V}(\mathcal{L})$ is a collection of vertices of a linkage $\mathcal{L}$, the *semiconfiguration space* is the set of restrictions to W of realizations of $\mathcal{L}$,

$$\mathcal{SC}(\mathcal{L}, W) = \{\, \varphi\colon W \to \mathbb{R}^n \text{ so that } \varphi = \varphi'|_W \text{ for some } \varphi' \in \mathcal{C}(\mathcal{L}) \,\}$$

Thus $\mathcal{SC}(\mathcal{L}, W)$ keeps track of the positions of only those vertices in W and ignores the positions of other vertices.

To characterize configuration and semiconfiguration spaces, we need a few definitions. A *real algebraic set* is the set of solutions of a collection of real polynomial equations in some $\mathbb{R}^m$. We define a *quasialgebraic set* to be a subset of $\mathbb{R}^m$ of the form

$$\{\, x \in \mathbb{R}^m \mid p_i(x) = 0,\ i = 1, \ldots, \ell \text{ and } q_j(x) \geq 0,\ j = 1, \ldots, k \,\}$$

for some polynomials p_i and q_j. Finally, a semialgebraic set is a finite union of differences of quasialgebraic sets. In other words, a semialgebraic set is a finite union of sets of the form

$$\{\, x \in \mathbb{R}^m \mid p_i(x) = 0, q_j(x) \geq 0, \text{ and } r_k(x) > 0 \,\}$$

for collections of polynomials p_i, q_j and r_k. Real algebraic sets and semialgebraic sets are well studied, but I am not aware of any literature on quasialgebraic sets.

We will use two notions of isomorphism. If $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^m$ then we say a homeomorphism $f\colon X \to Y$ is an *isomorphism* if f and f^{-1} are both restrictions of entire rational functions, for example polynomials. We say f is an *analytic isomorphism* if f and f^{-1} are both restrictions of analytic maps, i.e., maps locally given by power series. So any isomorphism is analytic, but the converse is not true. All analytic isomorphisms in this paper will actually be polynomials in one direction, but the inverse may involve square roots of positive quantities, and hence only be analytic.

Remark 1. Note that any configuration space $\mathcal{C}(\mathcal{L})$ of a classical linkage is an algebraic set in $(\mathbb{R}^n)^{\mathcal{V}(\mathcal{L})}$, since it is the solutions of the polynomial equations $|y_i - y_j|^2 = \ell_{ij}^2$ and $y_i = z_i$ for $i \in V$. Likewise, the configuration space of a cabled linkage is a quasialgebraic set in $(\mathbb{R}^n)^{\mathcal{V}(\mathcal{L})}$. Finally, a semiconfiguration space $\mathcal{SC}(\mathcal{L}, W)$ of a linkage $\mathcal{L}$ is a semialgebraic set in $(\mathbb{R}^n)^W$. To see this, note that it is the image of the quasialgebraic set $\mathcal{C}(\mathcal{L})$ under the projection map $(\mathbb{R}^n)^{\mathcal{V}(\mathcal{L})} \to (\mathbb{R}^n)^W$. But by the Tarski-Seidenberg theorem [6], the projection of a semialgebraic set is semialgebraic.

We will prove the following theorems characterizing configuration spaces of classical and cabled linkages, for $n \geq 3$. For $n = 2$, these theorems were proven in [3] and [1].

Theorem 1.1. *Suppose $\mathcal{L}$ is a classical linkage in $\mathbb{R}^n$, $n \geq 2$. Then $\mathcal{C}(\mathcal{L})$ is isomorphic to $X \times (\mathbb{R}^n)^k$ for some compact real algebraic set X. The integer k is the number of connected components of $\mathcal{L}$ with no fixed vertices.*

Conversely, if X is a compact real algebraic set and $k \geq 0$, there is a classical linkage $\mathcal{L}$ and a finite set F so that $\mathcal{C}(\mathcal{L})$ is analytically isomorphic to $X \times F \times (\mathbb{R}^n)^k$. In fact, there is an analytic function $\alpha \colon X \times F \times (\mathbb{R}^n)^k \to (\mathbb{R}^n)^m$ so that

$$\mathcal{C}(\mathcal{L}) = \{\, (x, \alpha(x, f, y)) \mid x \in X, y \in (\mathbb{R}^n)^k, \text{ and } f \in F \,\}$$

and so that the map $(x, y, f) \mapsto (x, \alpha(x, f, y))$ is an analytic isomorphism.

Theorem 1.2. *Suppose $\mathcal{L}$ is a cabled linkage in $\mathbb{R}^n$, $n \geq 2$. Then $\mathcal{C}(\mathcal{L})$ is isomorphic to $X \times (\mathbb{R}^n)^k$ for some compact quasialgebraic set X. The integer k is the number of connected components of $\mathcal{L}$ with no fixed vertices.*

Conversely, if X is a compact quasialgebraic set and $k \geq 0$, there is a cabled linkage $\mathcal{L}$ so that $\mathcal{C}(\mathcal{L})$ is analytically isomorphic to $X \times (\mathbb{R}^n)^k$. In fact, there is an analytic function $\alpha \colon X \times (\mathbb{R}^n)^k \to (\mathbb{R}^n)^m$ so that

$$\mathcal{C}(\mathcal{L}) = \{\, (x, \alpha(x, y)) \mid x \in X, \text{ and } y \in (\mathbb{R}^n)^k \,\}$$

and so that the map $(x, y) \mapsto (x, \alpha(x, y))$ is an analytic isomorphism.

We need a few more definitions before stating the characterization of semiconfiguration spaces, since that characterization is more precise.

Let $\mathrm{Euc}(n)$ denote the group of Euclidean motions of $\mathbb{R}^n$. So a general element of $\mathrm{Euc}(n)$ is of the form $z \mapsto Q(z) + z_0$ where $Q \in O(n)$ is an orthogonal matrix and $z_0 \in \mathbb{R}^n$ is a constant. We say a subset $Z \subset (\mathbb{R}^n)^k$ is *virtually compact* if either Z is compact, or Z is invariant under the diagonal action of $\mathrm{Euc}(n)$, with compact quotient.

Theorem 1.3. *Suppose $X \subset (\mathbb{R}^n)^k$, $n \geq 2$. Then the following are equivalent:*

1. *There is a cabled linkage $\mathcal{L}$ in $\mathbb{R}^n$ and a $W \subset \mathcal{V}(\mathcal{L})$ so that $\mathcal{SC}(\mathcal{L}, W) = X$.*
2. *There is a classical linkage $\mathcal{L}$ in $\mathbb{R}^n$ and a $W \subset \mathcal{V}(\mathcal{L})$ so that $\mathcal{SC}(\mathcal{L}, W) = X$.*
3. *After perhaps permuting the coordinates, $X = Y_1 \times Y_2 \times \cdots \times Y_m$ where each $Y_i \subset (\mathbb{R}^n)^{k_i}$ is a virtually compact semialgebraic set.*

Perhaps it is useful to restrict attention to the compact case. Then any compact semialgebraic set is the semiconfiguration space of a classical linkage. Any compact quasialgebraic set is analytically isomorphic to the configuration space of a cabled linkage. For any compact real algebraic set X there is a classical linkage whose configuration space is analytically isomorphic to a number of disjoint copies of X.

As a byproduct of the proof of Theorem 1.1, the cardinality of F will be 2^b. This comes about because there are b vertices which each have two distinct positioning modes. Essentially, Theorem 1.2 is proven by tethering these b vertices to fixed points so that only one of the two modes is possible (although other vertices are tethered as well).

Remark 2. If $\mathcal{L} = (L, \ell, V, \mu, F)$ is a linkage and $\beta \in \mathrm{Euc}(n)$ we may form a linkage $\beta(\mathcal{L}) = (L, \ell, V, \beta \circ \mu, F)$ by applying β to the image all fixed vertices. Since β preserves distances, we know that $\beta(\mathcal{C}(\mathcal{L})) = \mathcal{C}(\beta(\mathcal{L}))$ and $\beta(\mathcal{SC}(\mathcal{L}, W)) = \mathcal{SC}(\beta(\mathcal{L}), W)$. This is what we refer to as translating and rotating the linkage. We may also rescale $\mathcal{L}$ as follows. If λ is a positive number, then $\lambda\mathcal{L} = (L, \lambda\ell, V, \lambda\mu, F)$. We have $\lambda\mathcal{C}(\mathcal{L}) = \mathcal{C}(\lambda\mathcal{L})$ and $\lambda\mathcal{SC}(\mathcal{L}, W) = \mathcal{SC}(\lambda\mathcal{L}, W)$.

Because of the above remark, (semi)configuration spaces of linkages with few fixed points have a great deal of symmetry. We get more precise characterization theorems by taking this into account. While complete for semiconfiguration spaces, this characterization is incomplete for configuration spaces. For simplicity, we only state these results for connected linkages. Using Lemma 3.2 below, one could then formulate analogous results for nonconnected linkages.

To fix notation, let $e_i \in \mathbb{R}^n$ denote the unit vector whose only nonzero component is a 1 in the i-th place. For $1 \leq k \leq n$ we have a subgroup $O(k) \subset \mathrm{Euc}(n)$ which we fix on as the set of $\beta \in \mathrm{Euc}(n)$ so that $\beta(0) = 0$ and $\beta(e_i) = e_i$ for all $i \leq n - k$. We let $\mathrm{Tran}(n) \subset \mathrm{Euc}(n)$ denote the subgroup of translations, maps of the form $z \mapsto z + z_0$.

Theorem 1.4. *Suppose $\mathcal{L}$ is a connected linkage with exactly m fixed vertices and $W \subset \mathcal{V}(\mathcal{L})$, then:*

1. *If $m = 0$ then $\mathcal{SC}(\mathcal{L}, W)$ and $\mathcal{C}(\mathcal{L})$ are invariant under the action of* $\mathrm{Euc}(n)$, *with compact quotient.*
2. *If $1 \leq m \leq n$, then $\mathcal{SC}(\mathcal{L}, W)$ and $\mathcal{C}(\mathcal{L})$ are compact and invariant under a subgroup of* $\mathrm{Euc}(n)$ *conjugate to $O(n - m + 1)$.*
3. *To make part 2 above sharper, suppose $m \geq 1$ and T is an affine subspace of $\mathbb{R}^n$ which contains the images of all fixed vertices. Then $\mathcal{SC}(\mathcal{L}, W)$ and $\mathcal{C}(\mathcal{L})$ are compact and invariant under the subgroup of elements* $\mathrm{Euc}(n)$ *which fix all points of T. (This subgroup is conjugate to $O(n - \dim T)$.)*

Theorem 1.5. *Let $Z \subset (\mathbb{R}^n)^k$ be a virtually compact semialgebraic set, $n \geq 2$, and suppose that Z is invariant under the diagonal action of a subgroup G of* $\mathrm{Euc}(n)$, *with compact quotient. Then there is a connected classical linkage $\mathcal{L}$ and a $W \subset \mathcal{V}(\mathcal{L})$ so that $\mathcal{SC}(\mathcal{L}, W) = Z$ and so that*

1. *If $G =$* $\mathrm{Euc}(n)$, *then $\mathcal{L}$ has no fixed vertex.*
2. *If G is conjugate to $O(m)$, $1 \leq m \leq n$, then $\mathcal{L}$ has $n - m + 1$ fixed vertices, and these vertices are fixed at points on the fixed subspace of G.*
3. *Otherwise, $\mathcal{L}$ has only $n + 1$ fixed vertices.*

Theorem 1.6. *Let $Z \subset (\mathbb{R}^n)^k$ be a compact algebraic set, $n \geq 2$, and suppose that Z is invariant under the diagonal action of a subgroup G of* $\mathrm{Euc}(n)$. *Then there is a connected classical linkage $\mathcal{L}$ and a finite set F so that $\mathcal{C}(\mathcal{L})$ is analytically isomorphic to $Z \times F$ and so that*

1. *If G is conjugate to $O(1)$, then $\mathcal{L}$ has n fixed vertices, and these vertices are fixed at points on the fixed subspace of G.*
2. *Suppose G is conjugate to $O(2)$ and there is an algebraic subvariety $Z' \subset Z$ so that the map $(z, g) \mapsto gz$ from $Z' \times G^+$ to Z is an isomorphism, where $G^+ \subset G$ is the subgroup of orientation preserving elements of G. Then $\mathcal{L}$ has $n - 1$ fixed vertices, and these vertices are fixed at points in the fixed subspace of G.*

3. *Otherwise, $\mathcal{L}$ has only $n+1$ fixed vertices, which we may take to be fixed at 0 and e_i, $i = 1, \dots, n$.*

Moreover, there is an analytic function $\alpha\colon Z \times F \to (\mathbb{R}^n)^m$ so that

$$\mathcal{C}(\mathcal{L}) = \{\, (x, \alpha(x, f)) \mid x \in Z, \text{ and } f \in F \,\}$$

and so that the map $(x, f) \mapsto (x, \alpha(x, f))$ is an analytic isomorphism.

What is more, there is a converse to part 2 above if $n = 2$. If $\mathcal{L}$ is a connected planar classical linkage with 1 fixed vertex and at least one other vertex, then there is an algebraic subset $Z' \subset \mathcal{C}(\mathcal{L})$ and a subgroup $G^+ \subset \mathrm{Euc}(2)$ conjugate to $SO(2)$ so that the diagonal action $Z' \times G^+ \to \mathcal{C}(\mathcal{L})$ is an isomorphism.

Remark 3. For $n > 2$ and $G = O(2)$, the characterization of configuration spaces must be more complicated than that suggested by part 2 of Theorem 1.6. Consider for example the linkage in $\mathbb{R}^3$ with three vertices A, B, and C, and edge AC of length 1, and with A fixed at 0 and B fixed at e_1. Then $\mathcal{C}(\mathcal{L}) = 0 \times e_1 \times S^2$ which does not satisfy the hypotheses of part 2 of Theorem 1.6.

In the planar case $n = 2$ we may complete our description of configuration spaces by extending Theorem 1.6 to the noncompact case. We let $\mathrm{Euc}(n)^+$ denote the subgroup of $\mathrm{Euc}(n)$ consisting of orientation preserving Euclidean motions.

Theorem 1.7. *Suppose $Z \subset (\mathbb{R}^2)^k$ is an algebraic set invariant under the action of* $\mathrm{Euc}(2)$*. Suppose there is a compact algebraic subset $Z' \subset Z$ so that the map $\beta\colon Z' \times \mathrm{Euc}(2)^+ \to Z$ is an isomorphism where $\beta(z, g) = gz$. Then there is a connected classical linkage $\mathcal{L}$ with no fixed vertices and a finite set F so that $\mathcal{C}(\mathcal{L})$ is analytically isomorphic to $Z \times F$. Moreover, there is an analytic function $\alpha\colon Z \times F \to (\mathbb{R}^2)^m$ so that*

$$\mathcal{C}(\mathcal{L}) = \{\, (x, \alpha(x, f)) \mid x \in Z, \text{ and } f \in F \,\}$$

and so that the map $(x, f) \mapsto (x, \alpha(x, f))$ is an analytic isomorphism.

Conversely, if $\mathcal{L}$ is a connected classical planar linkage with no fixed vertices and at least two vertices then there is a compact algebraic subset $Z' \subset \mathcal{C}(\mathcal{L})$ so that the map $(z, g) \mapsto gz$ gives an isomorphism from $Z' \times \mathrm{Euc}(2)^+$ to $\mathcal{C}(\mathcal{L})$.

2. Functional Linkages

An essential ingredient in the proofs of the above theorems is the notion of a functional linkage. A (quasi)functional linkage is a linkage which "computes" some function.

A linkage $\mathcal{L}$ is *quasifunctional* for a map $f\colon (\mathbb{R}^n)^k \to (\mathbb{R}^n)^m$ if there are vertices $w_1, \dots, w_k$ and $v_1, \dots, v_m$ of $\mathcal{L}$ so that if $p\colon \mathcal{C}(\mathcal{L}) \to (\mathbb{R}^n)^m$ is $p(\varphi) = (\varphi(v_1), \dots, \varphi(v_m))$ and $q\colon \mathcal{C}(\mathcal{L}) \to (\mathbb{R}^n)^k$ is $q(\varphi) = (\varphi(w_1), \dots, \varphi(w_k))$ then $p = f \circ q$. The set $q(\mathcal{C}(\mathcal{L}))$ is called the *domain* of the quasifunctional linkage. We call q the input map and call p the output map.

If in addition, there is a $U \subset q(\mathcal{C}(\mathcal{L}))$ so that the restriction $q|\colon q^{-1}(U) \to U$ is an analytically trivial covering map, we say that $\mathcal{L}$ is *functional* for f with *restricted domain* U.

Moreover, if $q\colon \mathcal{C}(\mathcal{L}) \to q(\mathcal{C}(\mathcal{L}))$ is an analytic isomorphism we say that $\mathcal{L}$ is *strongly functional* for f. In this case, for expository convenience, if $U \subset q(\mathcal{C}(\mathcal{L}))$ we say that $\mathcal{L}$ is strongly functional for f with restricted domain U.

We call $w_1, \dots, w_k$ the *input vertices* and call $v_1, \dots, v_m$ the *output vertices*. Repetitions of vertices are allowed, although they are not necessary for the results in this paper.

So if $\mathcal{L}$ is functional for f, then over U the configuration space is a bunch of copies of the graph of f. If the configuration space is just one copy of the graph of f it is strongly functional.

The following is a key to the proofs of the above theorems. Its proof will occupy a substantial part of this paper.

Theorem 2.1. *Suppose $f\colon (\mathbb{R}^n)^k \to (\mathbb{R}^n)^m$ is a polynomial map and $K \subset (\mathbb{R}^n)^k$ is compact, $n \geq 2$. Then there is a functional classical linkage $\mathcal{L}$ for f with restricted domain K. There is also a strong functional cabled linkage $\mathcal{L}'$ for f with restricted domain K. We may specify that all input and output vertices of the functional linkages $\mathcal{L}$ and $\mathcal{L}'$ be distinct.*

3. Functoriality of $\mathcal{C}(\mathcal{L})$ and $\mathcal{SC}(\mathcal{L}, W)$

Let $\mathcal{L}' \subset \mathcal{L}$ be a sublinkage. This means that $L' \subset L$, $\ell' = \ell|_{\mathcal{E}(L')}$, $V' \subset V$, $\mu' = \mu|_{V'}$, and $F' = F \cap \mathcal{E}(L')$. Then we have a natural map $\rho_{\mathcal{L},\mathcal{L}'}\colon \mathcal{C}(\mathcal{L}) \to \mathcal{C}(\mathcal{L}')$ obtained by restriction, i.e., $\rho_{\mathcal{L},\mathcal{L}'}(\varphi) = \varphi|_{\mathcal{V}(L')}$. If L' is a single vertex v of $\mathcal{L}$ and has no edges and V' is empty, then we denote $\rho_{\mathcal{L},\mathcal{L}'} = \rho_{\mathcal{L},v}$. Thus $\rho_{\mathcal{L},v}(\varphi) = \varphi(v) \in \mathbb{R}^n = \mathcal{C}(\mathcal{L}')$.

If $\mathcal{L}' \subset \mathcal{L}$ and $\mathcal{L}'' \subset \mathcal{L}$ are two sublinkages then we may define their union $\mathcal{L}' \cup \mathcal{L}''$ as the sublinkage $(L''', \ell''', V''', \mu''', F''')$ of $\mathcal{L}$ with $L''' = L' \cup L''$ and $V''' = V' \cup V''$. Similarly, we may define the intersection $\mathcal{L}' \cap \mathcal{L}''$.

Lemma 3.1. *If $\mathcal{L}' \subset \mathcal{L}$ and $\mathcal{L}'' \subset \mathcal{L}$ are two sublinkages then we have a natural identification of $\mathcal{C}(\mathcal{L}' \cup \mathcal{L}'')$ with the fiber product of the restriction maps $\rho_{\mathcal{L}',\mathcal{L}'\cap\mathcal{L}''}\colon \mathcal{C}(\mathcal{L}') \to \mathcal{C}(\mathcal{L}' \cap \mathcal{L}'')$ and $\rho_{\mathcal{L}'',\mathcal{L}'\cap\mathcal{L}''}\colon \mathcal{C}(\mathcal{L}'') \to \mathcal{C}(\mathcal{L}' \cap \mathcal{L}'')$.*

$$\begin{array}{ccc}
\mathcal{C}(\mathcal{L}' \cup \mathcal{L}'') & \longrightarrow & \mathcal{C}(\mathcal{L}'') \\
\big\downarrow & & \big\downarrow \rho_{\mathcal{L}'',\mathcal{L}'\cap\mathcal{L}''} \\
\mathcal{C}(\mathcal{L}') & \xrightarrow{\rho_{\mathcal{L}',\mathcal{L}'\cap\mathcal{L}''}} & \mathcal{C}(\mathcal{L}' \cap \mathcal{L}'')
\end{array}$$

Proof. This is because a realization of $\mathcal{L}' \cup \mathcal{L}''$ is just a realization of $\mathcal{L}'$ and a realization of $\mathcal{L}''$ which happen to agree on $\mathcal{L}' \cap \mathcal{L}''$. Thus

$$\mathcal{C}(\mathcal{L}' \cup \mathcal{L}'') = \{(\varphi', \varphi'') \in \mathcal{C}(\mathcal{L}') \times \mathcal{C}(\mathcal{L}'') \mid \rho_{\mathcal{L}',\mathcal{L}'\cap\mathcal{L}''}(\varphi') = \rho_{\mathcal{L}',\mathcal{L}'\cap\mathcal{L}''}(\varphi'')\} \tag{3.1}$$

is the fiber product. Strictly speaking, rather than equality in equation (3.1) above, the map $(\rho_{\mathcal{L}'\cup\mathcal{L}'',\mathcal{L}'}\ ,\ \rho_{\mathcal{L}'\cup\mathcal{L}'',\mathcal{L}''})$ gives an isomorphism between the the two sides of equation (3.1). But we will suppress such distinctions. □

As a consequence of Lemma 3.1, the (semi)configuration space of the disjoint union of linkages is the product of their (semi)configuration spaces.

Lemma 3.2. *If $\mathcal{L}$ is the disjoint union of sublinkages $\mathcal{L}_i$, $i = 1, \dots, m$, then*

$$\begin{aligned}
\mathcal{C}(\mathcal{L}) &= \prod_{i=1}^{m} \mathcal{C}(\mathcal{L}_i) \\
\mathcal{SC}(\mathcal{L}, W) &= \prod_{i=1}^{m} \mathcal{SC}(\mathcal{L}_i, W \cap \mathcal{V}(\mathcal{L}_i))
\end{aligned}$$

Lemma 3.3. *Let $\mathcal{L}$ be a cabled linkage and let $G \subset \mathrm{Euc}(n)$ be a subgroup which fixes the images of all fixed vertices of $\mathcal{L}$. Then $\mathcal{C}(\mathcal{L})$ and $\mathcal{SC}(\mathcal{L}, W)$ are invariant under the diagonal action of G. In particular, if $\mathcal{L}$ has no fixed vertices, then $\mathcal{C}(\mathcal{L})$ and $\mathcal{SC}(\mathcal{L}, W)$ are invariant under the action of* $\mathrm{Euc}(n)$.

Proof. If $\beta \in G$, then $\beta(\mathcal{L}) = \mathcal{L}$. So $\beta(\mathcal{C}(\mathcal{L})) = \mathcal{C}(\beta(\mathcal{L})) = \mathcal{C}(\mathcal{L})$ and $\beta(\mathcal{SC}(\mathcal{L}, W)) = \mathcal{SC}(\beta(\mathcal{L}), W) = \mathcal{SC}(\mathcal{L}, W)$. □

Lemma 3.4. *Suppose $\mathcal{L}$ is a linkage with no fixed vertices. Form $\mathcal{L}'$ from $\mathcal{L}$ by fixing one of the vertices of $\mathcal{L}$ to some point. Then there is an isomorphism $\eta\colon \mathcal{C}(\mathcal{L}') \times \mathbb{R}^n \to \mathcal{C}(\mathcal{L})$ where $\eta(\varphi, z)(v) = \varphi(v) + z$ for any vertex v of $\mathcal{L}$.*

Proof. Translation preserves all lengths, so $\eta(\varphi, z) \in \mathcal{C}(\mathcal{L})$. But if $\mathcal{L}'$ is obtained by fixing a vertex v_0 to z_0, then the inverse of η is given by $\eta^{-1}(\varphi) = (\psi, z)$ where $\psi(v) = \varphi(v) - \varphi(v_0) + z_0$ and $z = \varphi(v_0) - z_0$. □

Lemma 3.5. *Let $\mathcal{L}$ be a cabled linkage and let $v_1, \dots, v_m$ be vertices of $\mathcal{L}$ which are not fixed. Let $\mathcal{L}'$ be obtained from $\mathcal{L}$ by fixing the vertices $v_1, \dots, v_m$ to be at the points $z_1, \dots, z_m$. Let $p\colon \mathcal{C}(\mathcal{L}) \to (\mathbb{R}^n)^m$ be the map $(\rho_{\mathcal{L},v_1}, \dots, \rho_{\mathcal{L},v_m})$. Then*

$$\mathcal{C}(\mathcal{L}') = p^{-1}(z_1, \dots, z_m)$$

If $W \subset \mathcal{V}(\mathcal{L})$ and $v_i \in W$ for all i, and $q\colon \mathcal{SC}(\mathcal{L}, W) \to (\mathbb{R}^n)^m$ is the map $q(\varphi) = (\varphi(v_1), \dots, \varphi(v_m))$. Then

$$\mathcal{SC}(\mathcal{L}', W) = q^{-1}(z_1, \dots, z_m)$$

Proof. Note that $\mathcal{C}(\mathcal{L}') \subset \mathcal{C}(\mathcal{L})$ and it must be exactly those φ with $\rho_{\mathcal{L},v_i}(\varphi) = \varphi(v_i) = z_i$. The lemma follows. □

Lemma 3.6. *Let $\mathcal{L}$ be a cabled linkage and let $v_1, \dots, v_m$ be vertices of $\mathcal{L}$. Let $\mathcal{L}'$ be obtained from $\mathcal{L}$ by adding new vertices $u_1, \dots, u_m$ and new flexible edges $\overline{u_i v_i}$ of length b_i. Fix the vertices u_i to points $z_i \in \mathbb{R}^n$. Let $p\colon \mathcal{C}(\mathcal{L}) \to (\mathbb{R}^n)^m$ be the map $(\rho_{\mathcal{L},v_1}, \dots, \rho_{\mathcal{L},v_m})$. Then $\mathcal{C}(\mathcal{L}')$ is isomorphic to*

$$p^{-1}(\{w \in (\mathbb{R}^n)^m \mid b_i \geq |w_i - z_i|,\ i = 1, \dots, m\})$$

Proof. The inclusion $\mathcal{L} \subset \mathcal{L}'$ gives the map $\rho_{\mathcal{L}',\mathcal{L}}\colon \mathcal{C}(\mathcal{L}') \to \mathcal{C}(\mathcal{L})$. Let

$$Y = p^{-1}(\{w \in (\mathbb{R}^n)^m \mid b_i \geq |w_i - z_i|,\ i = 1, \dots, m\})$$

We have a map $\beta\colon Y \to \mathcal{C}(\mathcal{L}')$ defined by $\beta(\varphi)(v) = \varphi(v)$ for v a vertex of $\mathcal{L}$ and $\beta(\varphi)(u_i) = z_i$. Note that $\rho_{\mathcal{L}',\mathcal{L}}(\mathcal{C}(\mathcal{L}')) \subset Y$ and β is the inverse of $\rho_{\mathcal{L}',\mathcal{L}}\colon \mathcal{C}(\mathcal{L}') \to Y$. □

The following two lemmas are immediate from the definitions.

Lemma 3.7. *If $\mathcal{L}' \subset \mathcal{L}$ is a sublinkage then the map $\rho_{\mathcal{L},\mathcal{L}'}\colon \mathcal{C}(\mathcal{L}) \to \mathcal{C}(\mathcal{L}')$ is an (analytic) isomorphism if and only if it is onto and the position $\varphi(v)$ of each vertex v of $\mathcal{L}$ is a rational (resp. analytic) function of the positions $\varphi(w_i)$ of the vertices w_i in $\mathcal{L}'$. More generally, if $Z \subset \mathcal{C}(\mathcal{L})$ then the restriction $\rho_{\mathcal{L},\mathcal{L}'}|_Z\colon Z \to \rho_{\mathcal{L},\mathcal{L}'}(Z)$ is an (analytic) isomorphism if and only if for $\varphi \in Z$, the position $\varphi(v)$ of each vertex v of $\mathcal{L}$ is a rational (resp. analytic) function of the positions $\varphi(w_i)$ of the vertices w_i in $\mathcal{L}'$.*

Lemma 3.8. *Let $\mathcal{L}$ be a linkage and suppose v and w are two vertices of $\mathcal{L}$. Suppose that whenever there are edges $\overline{vu}$ and $\overline{wu}$ to the same vertex u, that $\ell(\overline{vu}) = \ell(\overline{vw})$. Suppose also that there is no edge $\overline{vw}$. Then we may form a linkage $\mathcal{L}'$ from $\mathcal{L}$ by identifying the vertices v and w, and identifying any edges $\overline{vu}$ and $\overline{wu}$. Moreover there is a natural identification of $\mathcal{C}(\mathcal{L}')$ with $\{\varphi \in \mathcal{C}(\mathcal{L}) \mid \varphi(v) = \varphi(w)\}$.*

Note in Lemma 3.8 that if $\{\varphi \in \mathcal{C}(\mathcal{L}) \mid \varphi(v) = \varphi(w)\}$ is nonempty, then the hypotheses of Lemma 3.8 must be satisfied.

Lemma 3.9. *If $\mathcal{SC}(\mathcal{L}, W)$ is compact and nonempty, then there is a path in L from each point of W to a vertex which is fixed.*

Proof. By Lemma 3.2, it suffices to assume that $\mathcal{L}$ is connected. If $\mathcal{L}$ has no fixed vertices, we may translate any realization to get another realization. So $\mathcal{SC}(\mathcal{L}, W)$ would be noncompact, c.f. Lemma 3.4. So $\mathcal{L}$ must have a fixed vertex, and by connectedness there is a path from each vertex of W to this fixed vertex. □

4. Constructing Polynomial Functional Linkages

Simple modifications of two functional linkages allow us to construct a functional linkage for their composition or cartesian product.

Lemma 4.1. *Let $\mathcal{L}$ and $\mathcal{L}'$ be (strong) functional linkages for functions $f: (\mathbb{R}^n)^k \to (\mathbb{R}^n)^m$ and $g: (\mathbb{R}^n)^m \to (\mathbb{R}^n)^\ell$ with restricted domains U and U'. Suppose that $U \cap f^{-1}(U')$ is nonempty. We may form a linkage $\mathcal{L}''$ by taking the disjoint union of $\mathcal{L}$ and $\mathcal{L}'$ and then identifying each output vertex of $\mathcal{L}$ with the corresponding input vertex of $\mathcal{L}'$. Then $\mathcal{L}''$ is a (strong) functional linkage for $g \circ f$ with restricted domain $U \cap f^{-1}(U')$.*

Proof. Let $v_1, \dots, v_m$ be the input vertices of $\mathcal{L}'$ and let $w_1, \dots, w_m$ be the output vertices of $\mathcal{L}$. Suppose first that any duplications in the v_i correspond to duplications in the w_i and vice versa. So $v_i = v_j$ if and only if $w_i = w_j$. Then $\mathcal{L}''$ is the union of $\mathcal{L}$ and $\mathcal{L}'$, and their intersection is the linkage with no edges and with vertices $V = \{v_1, v_2, \dots, v_m\}$. Let ρ_1 and ρ_2 be the input and output maps of $\mathcal{L}$ and let ρ_3 and ρ_4 be the input and output maps of $\mathcal{L}'$. By Lemma 3.1, we know that $\mathcal{C}(\mathcal{L}'')$ is the fiber product of $\rho_{\mathcal{L},\mathcal{L}\cap\mathcal{L}'}$ and $\rho_{\mathcal{L}',\mathcal{L}\cap\mathcal{L}'}$. If $q: \mathcal{C}(\mathcal{L}\cap\mathcal{L}') \to (\mathbb{R}^n)^m$ is the map $q = (\rho_{\mathcal{L}\cap\mathcal{L}',v_1}, \dots, \rho_{\mathcal{L}\cap\mathcal{L}',v_m})$, then $\rho_2 = q \circ \rho_{\mathcal{L},\mathcal{L}\cap\mathcal{L}'}$ and $\rho_3 = q \circ \rho_{\mathcal{L}',\mathcal{L}\cap\mathcal{L}'}$. If $v_1, \dots, v_m$ are all distinct, q will be the identity, but if there are duplications, q will be some sort of diagonal map. Since q is injective, $\mathcal{C}(\mathcal{L}'')$ is also the fiber product of ρ_2 and ρ_3,

$$\mathcal{C}(\mathcal{L}'') = \{(\varphi, \varphi') \in \mathcal{C}(\mathcal{L}) \times \mathcal{C}(\mathcal{L}') \mid \rho_2(\varphi) = \rho_3(\varphi')\} \tag{4.1}$$

so that $\rho_{\mathcal{L}'',\mathcal{L}}$ and $\rho_{\mathcal{L}'',\mathcal{L}'}$ are induced by projection. Note that

$$g \circ f \circ \rho_1 \circ \rho_{\mathcal{L}'',\mathcal{L}} = g \circ \rho_2 \circ \rho_{\mathcal{L}'',\mathcal{L}} = g \circ \rho_3 \circ \rho_{\mathcal{L}'',\mathcal{L}'} = \rho_4 \circ \rho_{\mathcal{L}'',\mathcal{L}'}$$

so $\mathcal{L}''$ is quasifunctional for $g \circ f$. The input map is $\rho_1 \circ \rho_{\mathcal{L}'',\mathcal{L}}$ and the output map is $\rho_4 \circ \rho_{\mathcal{L}'',\mathcal{L}'}$.

Now let us see that we can take the restricted domain to be $U \cap f^{-1}(U')$. The restriction of ρ_1 to $\rho_1^{-1}(U \cap f^{-1}(U'))$ is an analytically trivial cover since the restriction to $\rho_1^{-1}(U)$ is, so we only need show that $\rho_{\mathcal{L}'',\mathcal{L}}$ restricts to an analytically trivial cover of $\rho_1^{-1}(U \cap f^{-1}(U')) = \rho_1^{-1}(U) \cap \rho_2^{-1}(U')$. We know that there is a finite set F and an analytic isomorphism $\sigma\colon U' \times F \to \rho_3^{-1}(U')$ so that $\rho_3 \circ \sigma$ is projection to U'. Now by (4.1), we have

$$\begin{aligned} \rho_{\mathcal{L}'',\mathcal{L}}^{-1}(\rho_2^{-1}(U')) &= \{(\varphi,\varphi') \mid \rho_2(\varphi) = \rho_3(\varphi') \in U'\} \\ &= \{(\varphi, \sigma(\rho_2(\varphi), c)) \mid \rho_2(\varphi) \in U' \text{ and } c \in F\} \end{aligned}$$

So we have an analytic trivialization $\sigma'\colon \rho_2^{-1}(U') \times F \to \rho_{\mathcal{L}'',\mathcal{L}}^{-1}(\rho_2^{-1}(U'))$ given by $\sigma'(\varphi, c) = (\varphi, \sigma(\rho_2(\varphi), c))$. So $\mathcal{L}''$ is functional for $g \circ f$ with restricted domain $U \cap f^{-1}(U')$.

To prove the strong case, note that all the covers are one-fold and hence are analytic isomorphisms and we may take $U = \rho_1(\mathcal{C}(\mathcal{L}))$ and $U' = \rho_3(\mathcal{C}(\mathcal{L}'))$.

If there are duplications in the input and output vertices things can get more complicated, since we may end up having to identify vertices in $\mathcal{L}$ or $\mathcal{L}'$ which were not previously identified. Let $\Delta_{ij} = \{(z_1, \ldots, z_m) \in (\mathbb{R}^n)^m \mid z_i = z_j\}$.

Suppose $v_i = v_j$ but $w_i \neq w_j$. Then we must have $U' \subset \Delta_{ij}$. Also, in $\mathcal{L}''$ we end up identifying w_i with w_j. Let us first see whether we can do so according to Lemma 3.8. Suppose w is another vertex so that $\overline{ww_i}$ and $\overline{ww_j}$ are both edges of $\mathcal{L}$. Since $U \cap f^{-1}(U')$ is nonempty, there is a $\varphi \in \mathcal{C}(\mathcal{L})$ so that $\rho_1(\varphi) \in U \cap f^{-1}(U')$. Hence $\rho_2(\varphi) = f(\rho_1(\varphi)) \in U' \subset \Delta_{ij}$, and so $\varphi(w_i) = \varphi(w_j)$. So

$$\ell(\overline{ww_i}) = |\varphi(w) - \varphi(w_i)| = |\varphi(w) - \varphi(w_j)| = \ell(\overline{ww_j})$$

So in $\mathcal{L}''$ we may identify the edges $\overline{ww_i}$ and $\overline{ww_j}$ since they have the same length. There could not be an edge $\overline{w_iw_j}$ since $0 \neq \ell(\overline{w_iw_j}) = |\varphi(w_i) - \varphi(w_j)| = 0$. So by Lemma 3.8 we are allowed to take the quotient linkage $\mathcal{L}_1$ of $\mathcal{L}$, identifying w_i and w_j. By Lemma 3.8 we also see that $\mathcal{L}_1$ is still functional for f but the domain has shrunk from $\rho_1(\mathcal{C}(\mathcal{L}))$ to $\rho_1(\mathcal{C}(\mathcal{L})) \cap f^{-1}(\Delta_{ij})$. So we may take the restricted domain of $\mathcal{L}_1$ to be $U \cap f^{-1}(\Delta_{ij})$. Do this identification for each pair i, j with $v_i = v_j$ and $w_i \neq w_j$ and we eventually get a functional linkage $\mathcal{L}_2$ for f with restricted domain $U_2 = U \cap f^{-1}(\Delta)$ for some $\Delta \supset U'$.

Now suppose $w_i = w_j$, but $v_i \neq v_j$. Then we must have $f(U) \subset \Delta_{ij}$. Also, in $\mathcal{L}''$ we end up identifying v_i with v_j. Let us see whether we can do so. Suppose v is another vertex so that $\overline{vv_i}$ and $\overline{vv_j}$ are both edges of $\mathcal{L}'$. Since $U \cap f^{-1}(U')$ is nonempty, we know $\Delta_{ij} \cap U'$ is nonempty, so there is a $\varphi \in \mathcal{C}(\mathcal{L}')$ so that $\rho_3(\varphi) \in \Delta_{ij} \cap U'$, and hence $\varphi(v_i) = \varphi(v_j)$. So as above, Lemma 3.8 will allow us to take the quotient linkage $\mathcal{L}_1'$ identifying v_i and v_j. By Lemma 3.8 we also see that $\mathcal{L}_1'$ is still functional for g but with restricted domain $U' \cap \Delta_{ij}$. Do this identification for each pair i, j with $w_i = w_j$ and $v_i \neq v_j$ and we eventually get a functional linkage $\mathcal{L}_2'$ for g with restricted domain $U_2' = U' \cap \Delta'$ for some $\Delta' \supset f(U)$.

After doing all these identifications, we have $\mathcal{L}''$ is the union of $\mathcal{L}_2$ and $\mathcal{L}_2'$, and we may finish the proof as above. The only thing to check is that $U_2 \cap f^{-1}(U_2') = U \cap f^{-1}(U')$. But $U_2 \cap f^{-1}(U_2') = U \cap f^{-1}(\Delta) \cap f^{-1}(U') \cap f^{-1}(\Delta') = U \cap f^{-1}(U')$ since $U' \subset \Delta$ and $U \subset f^{-1}(\Delta')$. □

Lemma 4.2. *For $i = 0, 1$, let $\mathcal{L}_i$ be (strong) functional linkages for functions $f_i\colon (\mathbb{R}^n)^{k_i} \to (\mathbb{R}^n)^{m_i}$ with restricted domain U_i. Form a linkage $\mathcal{L}$ by taking the disjoint union of $\mathcal{L}_0$ and $\mathcal{L}_1$. If $k_0 = k_1$, form a linkage $\mathcal{L}'$ by taking $\mathcal{L}$ and identifying corresponding input vertices of $\mathcal{L}_0$ and $\mathcal{L}_1$.*

1. *Then $\mathcal{L}$ is a (strong) functional linkage for*

$$f_0 \times f_1\colon (\mathbb{R}^n)^{k_0} \times (\mathbb{R}^n)^{k_1} \to (\mathbb{R}^n)^{m_0} \times (\mathbb{R}^n)^{m_1}$$

with restricted domain $U_0 \times U_1$. (In particular, if $m_1 = 0$ then $f_0 \times f_1$ is the composition of f_0 with projection $(\mathbb{R}^n)^{k_0} \times (\mathbb{R}^n)^{k_1} \to (\mathbb{R}^n)^{k_0}$.)

2. *If $k_0 = k_1$, $\mathcal{L}'$ is a (strong) functional linkage for*
$$(f_0, f_1)\colon (\mathbb{R}^n)^{k_0} \to (\mathbb{R}^n)^{m_0} \times (\mathbb{R}^n)^{m_1}$$
with restricted domain $U_0 \cap U_1$.

Proof. The statement for $\mathcal{L}$ is trivial to prove. If $k_0 = k_1 = k$, let $\mathcal{L}''$ be the linkage with no edges and k vertices $v_1, \dots, v_k$. If we let these vertices be the input vertices and let the output vertices be doubled, $v_1, \dots, v_k, v_1, \dots, v_k$, then we get a strong functional linkage for the diagonal map $\Delta(z) = (z, z)$ with domain all of $(\mathbb{R}^n)^k$. By Lemma 4.1, $\mathcal{L}'$ is the composition of the linkages $\mathcal{L}$ and $\mathcal{L}''$ and hence is (strongly) functional for $(f_0 \times f_1) \circ \Delta = (f_0, f_1)$. □

4.1. **Elementary Polynomial Functional Linkages.** We are now ready to make the first progress in proving Theorem 2.1. We first reduce it to finding (strong) functional linkages for addition, multiplication, and some linear maps.

Reduction 4.3. *To prove Theorem 2.1, it suffices to prove the existence of (strong) functional linkages for the following functions, each with arbitrarily large compact restricted domain, and distinct input and output vertices.*

1. *$q\colon (\mathbb{R}^n)^2 \to \mathbb{R}^n$ given by $q(x, y) = x + y$.*
2. *$r\colon T \times T \to T$ given by $r(sz_0, tz_0) = stz_0$, where $z_0 \neq 0$ and T is the line through 0 and z_0.*
3. *$u\colon \mathbb{R}^n \to \mathbb{R}^n$ any rank 1 linear transformation.*

Proof. Suppose $f\colon (\mathbb{R}^n)^k \to (\mathbb{R}^n)^m$ is a polynomial map for which we wish to find a (strong) functional linkage. By Lemma 4.2, it suffices to consider the case $m = 1$, i.e., of polynomials $f\colon (\mathbb{R}^n)^k \to \mathbb{R}^n$. Note that if $\mathcal{L}_i$ are functional linkages for $f_i\colon (\mathbb{R}^n)^k \to \mathbb{R}^n$, by Lemma 4.2 we may form a functional linkage for $(f_0, f_1)\colon (\mathbb{R}^n)^k \to \mathbb{R}^n \times \mathbb{R}^n$, then using Lemma 4.1 and composing with a functional linkage for $q(x, y) = x + y$ we get a functional linkage for $f_0 + f_1\colon (\mathbb{R}^n)^k \to \mathbb{R}^n$. So it suffices to find functional linkages for $f\colon (\mathbb{R}^n)^k \to \mathbb{R}^n$ of the form $f(x) = p(x)e_j$ for p a monomial. If $\mathcal{L}_i$ are functional linkages for $f_i\colon (\mathbb{R}^n)^k \to \mathbb{R}^n$, of the form $f_i(x) = p_i(x)e_j$, then by Lemma 4.2, Lemma 4.1 and the map r above, we get a functional linkage for the map $x \mapsto p_0(x)p_1(x)e_j$. So it suffices to find functional linkages for degree 0 or 1 monomials. Degree 1 monomials are linear and so are obtained from the map u above. In particular, we take a functional linkage for u and add $k - 1$ disjoint vertices which we designate as input vertices. Degree 0 monomials are constants, use a trivial linkage with one fixed output vertex and k input vertices. □

While the functions we reduced to in Reduction 4.3 seem natural, they are not all suited to easy description as linkages. So we make a further reduction to some elementary functions for which we can more readily provide linkages. It is interesting to note that by item 4 below, we could actually construct functional linkages for any rational function. Later in this paper we will investigate further exactly which functions admit functional linkages.

Reduction 4.4. *To prove Theorem 2.1, it suffices to prove the existence of (strong) functional linkages for the following functions, all with distinct input and output vertices.*

1. *Translation:* $z \mapsto z + z_0$*, with restricted domain any compact* $K \subset \mathbb{R}^n$*.*
2. *Scalar multiplication:* $z \mapsto \lambda z$*, with restricted domain a disc* $\{|z - z_0| \leq r\}$ *for some* z_0 *and for any* r *as large as we wish.*
3. *Average:* $(z, w) \mapsto (z + w)/2$*, with restricted domain* $\{(z, w) \mid |z - z_0| \leq r, |w + z_0| \leq r\}$ *for some* z_0 *and for any* r *as large as we wish.*
4. *Inversion in a line:* $sz_0 \mapsto (1/s)z_0$*, for any specified* $z_0 \in \mathbb{R}^n - 0$*, with restricted domain any compact* $K \subset \{tz_0 \mid t \neq 0\}$*.*
5. *Orthogonal projection to a line:* $z \mapsto (z \cdot z_0)z_0$*, for any unit vector* $z_0 \in \mathbb{R}^n$*, with restricted domain any compact* K*.*

Proof. Note first that we may always further restrict the domain of a functional linkage, so it suffices to find functional linkages with arbitrarily large compact restricted domains, for example (products of) balls of radius r. By Reduction 4.3, we only need to use the above five types of functional linkages to construct functional linkages for the three types of functions listed in Reduction 4.3.

But before we do this, we will show that for scalar multiplication 2 above, we may actually take the restricted domain to be an arbitrarily large ball $\{|z| \leq r\}$. First use 2 above to get a functional linkage for $z \mapsto \lambda z$, with restricted domain $\{|z - z_0| \leq r\}$ for some z_0. Then use 1 to get a functional linkage for translation $z \mapsto z + z_0$ with restricted domain $|z| \leq r$. Using Lemma 4.1, compose these two to get a functional linkage for $z \mapsto \lambda z + \lambda z_0$ with restricted domain $|z| \leq r$. Now using 1 and Lemma 4.1, compose with a translation by $-\lambda z_0$ to get our desired linkage for $z \mapsto \lambda z$ with restricted domain $|z| \leq r$.

To get $(z, w) \mapsto z + w$ with restricted domain $|z| \leq r$, $|w| \leq r$, find a functional linkage for the average 3 above, with restricted domain $\{(z, w) \mid |z - z_0| \leq r, |w + z_0| \leq r\}$ for some z_0. Then using 1, find functional linkages for $z \mapsto z + z_0$ and $z \mapsto z - z_0$, both with restricted domain $|z| \leq r$. By Lemma 4.2, their disjoint union is functional for $(z, w) \mapsto (z + z_0, w - z_0)$ with restricted domain $|z| \leq r$, $|w| \leq r$. Using Lemma 4.1 and composing with the first linkage, we get a functional linkage for $(z, w) \mapsto (z + w)/2$ with restricted domain $|z| \leq r$, $|w| \leq r$. Now composing with scalar multiplication by 2 with restricted domain $|z| \leq r$, we get a functional linkage for $(z, w) \mapsto z + w$ with restricted domain $|z| \leq r$, $|w| \leq r$.

Next we will find a (strong) functional linkage for any rank one linear map $u\colon \mathbb{R}^n \to \mathbb{R}^n$. But any rank one linear map is a composition of orthogonal projections to lines followed by scalar multiplication. So compositions of maps 5 and 2 will give us a (strong) functional linkage for u. (To see that u is such a composition, first do orthogonal projection to the orthogonal compliment of $\ker u$. If $u^2 \neq 0$, we may then orthogonal project to the image of u, and multiply by an appropriate scalar to get u. If $u^2 = 0$, do two more orthogonal projections, first to a line T which is neither perpendicular to the image of u nor contained in $\ker u$, and then orthogonal project to the image of u. Finally, multiply by an appropriate scalar.)

So the only remaining function is multiplication. We will do some algebraic manipulation to get multiplication $(sz_0, tz_0) \mapsto stz_0$. First, note that

$$st = ((s + t)^2 - (s - t)^2)/4$$

So it suffices to find a functional linkage for $sz_0 \mapsto s^2 z_0$ with restricted domain $\{\, |s| \leq r \,\}$. By 4 above there is a functional linkage for the map $h(sz_0') = (1/s)z_0'$ with restricted domain $2/3 \leq s \leq 4/3$, where $z_0' = 3rz_0$. But we note that

$$z_0' - h((h((1 + s)z_0') + h((1 - s)z_0'))/2) = s^2 z_0'$$

FIGURE 1. How to put a joint in the middle of an edge

so by composing known functional linkages, we get a functional linkage for the function $sz_0' \mapsto s^2 z_0'$ with restricted domain $|s| \leq 1/3$. But this is the same as the function $sz_0 \mapsto (s^2/(3r))z_0$ with restricted domain $|s| \leq r$. So after composing with multiplication by $3r$, we get the desired functional linkage. $\square$

So we have now reduced the proof of Theorem 2.1 to finding the functional linkages 1-5 in Reduction 4.4. In doing so, the following Lemma will be useful. Its proof may be safely left to the reader. It is, for example, a special case of the theorem that a proper submersion is a locally trivial fibration.

Lemma 4.5. *Let $f\colon M \to \mathbb{R}^n$ be a smooth map from a compact n dimensional manifold with boundary. Let $S \subset M$ be the set of critical points of f, the points where df has rank $< n$. Let U be any connected component of $\mathbb{R}^n - f(S \cup \partial M)$. Then the restriction $f|\colon f^{-1}(U) \to U$ is a covering projection.*

In our usage, f is analytic, and U is often contractible, so f restricts to an analytically trivial covering of U, thus $f^{-1}(U)$ is analytically isomorphic to $U\times$ a finite set. As another application, we will use the consequence that $f(M)$ is the union of $f(S \cup \partial M)$ and some connected components of $\mathbb{R}^n - f(S \cup \partial M)$.

So in the remainder of this section we will construct the functional linkages 1-5 listed in Reduction 4.4 above. But first we look at some useful examples.

4.2. **Simulating interior joints, cables, and telescoping edges.** In our model of linkages, edges are connected only at their ends. Actual linkages used in real life might have a connection in the middle of an edge. This may be simulated as in Figure 1. If $\ell(\overline{AB}) = b$ and we wish to place a connection C in the middle of $\overline{AB}$, let $\ell(\overline{AC}) = a$ and $\ell(\overline{BC}) = b - a$. Thus when drawing linkages, it is allowable to draw a joint in the middle of an edge.

If we are in the context of semiconfiguration spaces, we can also simulate other types of linkages. For example, suppose we want two vertices A and B connected by a cable, so the distance between them is constrained to be $\leq b$. More generally, suppose we wish to connect A and B by a telescoping edge, so the distance between them is constrained to be in the interval $[a, b]$. This can be simulated as in Figure 2. Since we are using semiconfiguration spaces, we can ignore the position of the vertex D. To simulate a cable, we take $c = d = b/2$. To simulate a telescoping edge with $0 < a < b$, we take $c = (a + b)/2$, $d = (b - a)/2$.

4.3. **The rigidified parallelogram.** When constructing linkages, one often wants four vertices to lie in a plane, and moreover to form a parallelogram. The linkage of Figure 3 will do this. There are six vertices A, B, C, D, E, F and nine edges $\overline{AB}$ and $\overline{CD}$ of length a, $\overline{AC}$, $\overline{BD}$, and $\overline{EF}$ of length b and $\overline{AE}$, $\overline{BE}$, $\overline{CF}$, and $\overline{DF}$ of length $a/2$.

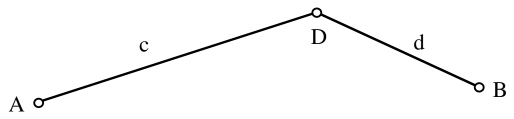

FIGURE 2. Simulating a cable or telescoping edge

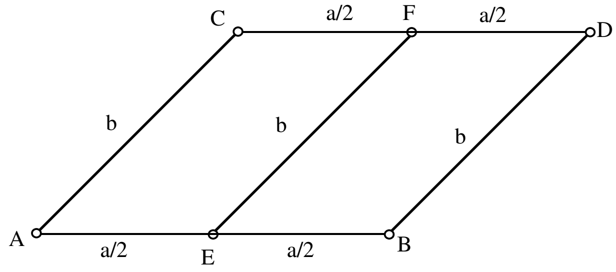

FIGURE 3. A rigidified parallelogram

Note that E and F are simulated interior joints, so for any realization φ, we must have $\varphi(E) = (\varphi(A) + \varphi(B))/2$ and $\varphi(F) = (\varphi(C) + \varphi(D))/2$.

We claim that $\varphi(A), \varphi(B), \varphi(C), \varphi(D)$ form a parallelogram. To see this, let $x = \varphi(B) - \varphi(A)$, $y = \varphi(C) - \varphi(A)$, and $z = \varphi(D) + \varphi(A) - \varphi(C) - \varphi(B)$. Then $\varphi(E) = \varphi(A) + x/2$, $\varphi(D) = \varphi(A) + x + y + z$, and $\varphi(F) = \varphi(A) + y + (x + z)/2$. The side length equations become $|x| = a$, $|x + z| = a$, $|y| = b$, $|y + z| = b$, and $|y + z/2| = b$. From the last three equations we see that $z = 0$ and hence that $\varphi(A), \varphi(B), \varphi(C), \varphi(D)$ form a parallelogram.

Henceforth, when drawing such a rigidified parallelogram we will draw the edge $\overline{EF}$ as a gray line in an attempt to unclutter the drawings. We will also usually refrain from naming the vertices E and F.

If $b = a$, we will often refer to this as a rigidified square, (although realizations usually do not have right angles). We take this terminology rigidified square or rigidified parallelogram from [3]. Note however that the linkage is not completely rigid, but retains some flexibility. In fact, the quotient space $\mathcal{C}(\mathcal{L})/Euc(n)$ is an interval, parameterized by the angle at a vertex. If the rigidifying edge $\overline{EF}$ were not present, the configuration space would be bigger, including configurations obtained by bending along $\overline{CB}$ or $\overline{AD}$ which one generally does not want.

4.4. **Making spheres.**

Lemma 4.6. *Given any round k-sphere $S \subset \mathbb{R}^n$, there is a classical linkage $\mathcal{L}$ and a vertex $v \in \mathcal{V}(\mathcal{L})$ so that $S = \mathcal{SC}(\mathcal{L}, \{v\})$. Moreover $\rho_{\mathcal{L},v} : \mathcal{C}(\mathcal{L}) \to \mathbb{R}^n$ is an isomorphism to its image S.*

Proof. After translation and rotation we may assume that

$$S = \{x \in \mathbb{R}^n \mid x_i = 0, i = 1, \ldots, n-k-1 \text{ and } |x| = r\}$$

Consider a linkage $\mathcal{L}$ with fixed vertices v_i, $i = 0, \ldots, n-k-1$ and one other vertex v, and edges $\overline{vv_i}$, $i = 0, \ldots, n-k-1$. We fix v_0 at 0 and fix v_i at the point re_i if $i \geq 1$. We let the length of $\overline{vv_0}$ be r, and let the length of $\overline{vv_i}$ be $\sqrt{2}r$ for $i \geq 1$.

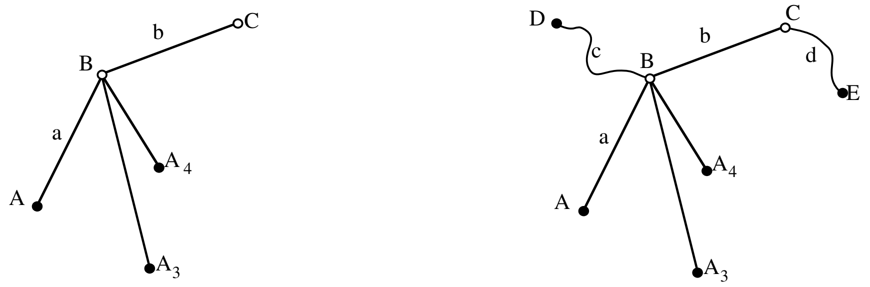

FIGURE 4. A Basic Linkage

Then if $\varphi \in \mathcal{C}(\mathcal{L})$ and $\varphi(v) = x$, we know that $r = \ell(\overline{vv_0}) = |x|$ and

$$2r^2 = |x - re_i|^2 = |x|^2 - 2rx_i + r^2 = 2r^2 - 2rx_i$$

for all $i = 1, \dots, n-k-1$ so we know that $x_i = 0$ for $i \leq n-k-1$, so $x \in S$. Conversely, if $x \in S$ then $x = \varphi(v)$ for some $\varphi \in \mathcal{C}(\mathcal{L})$.

The map $\rho_{\mathcal{L},v}$ is an isomorphism to S since v is the only vertex which is not fixed. □

4.5. **A simple Linkage, a key to understanding more complicated Linkages.** It will be useful to look first at a simple linkage $\mathcal{L}$, as shown in the left half of Figure 4. Using Lemma 4.6, we start with a linkage with $n-1$ fixed vertices, A and A_i, $i = 3, \dots, n$, and a movable vertex B, so that B is constrained to move in a circle with center A and radius a. We then add a final vertex C and an edge $\overline{BC}$ of length b. We assume that $b \leq a$.

To make some choices, we fix A at a point z_1, fix the A_i vertices at $z_1 + ae_i$ for $3 \leq i \leq n$, and put in edges $\overline{A_iB}$ of length $\sqrt{2}a$ and $\overline{AB}$ of length a. Then B is constrained to lie in a circle

$$S = \{z \in \mathbb{R}^n \mid (z - z_1) \cdot e_i = 0 \text{ for } i \geq 3 \text{ and } |z - z_1| = a\}$$

and C is only constrained to be in a sphere around B of radius b. Consequently $\mathcal{C}(\mathcal{L})$ is a torus $S^1 \times S^{n-1}$, where S^1 is the unit circle about the origin in the x_1x_2 plane and S^{n-1} is the unit sphere about the origin in $\mathbb{R}^n$. We may identify $(u, v) \in S^1 \times S^{n-1}$ with φ_{uv} where $\varphi_{uv}(A) = z_1$, $\varphi_{uv}(B) = z_1 + au$, and $\varphi_{uv}(C) = z_1 + au + bv$. Note that $\rho_{\mathcal{L},C} : \mathcal{C}(\mathcal{L}) \to \mathbb{R}^n$ is then the map $\rho_{\mathcal{L},C}(u, v) = z_1 + au + bv$ which has critical set where $v_1u_2 = u_1v_2$. The image of the critical set is the torus of points at distance b from the circle S. So by Lemma 4.5 we see that the image of $\rho_{\mathcal{L},C}$ is the solid torus T of points at distance $\leq b$ from S. Moreover $\rho_{\mathcal{L},C}$ restricts to a double cover of the interior of T. In fact this double cover is analytically trivial. In applications below, we will usually only focus on some disc $\{|z - z_0| \leq r\}$ inside the solid torus where, say, $z_0 \in S$ and $0 < r < b$. Then $\rho_{\mathcal{L},C}$ restricts to an analytically trivial double cover of this disc, (since it is an analytic proper submersion over the disc, and hence a locally analytically trivial fiber bundle).

When working with cabled linkages, we will want to modify this linkage so that $\rho_{\mathcal{L},C}$ is an analytic isomorphism to some disc $\{|z - z_0| \leq d\}$. We do this by tethering the vertices B and C to fixed vertices D and E so that their movement is restricted, see the cabled linkage on the right half of Figure 4. Consider first the sublinkage $\mathcal{L}'$ formed by A, A_i, B, C, and D, with rigid edges $\overline{AB}$, $\overline{A_iB}$ and $\overline{BC}$, and a flexible

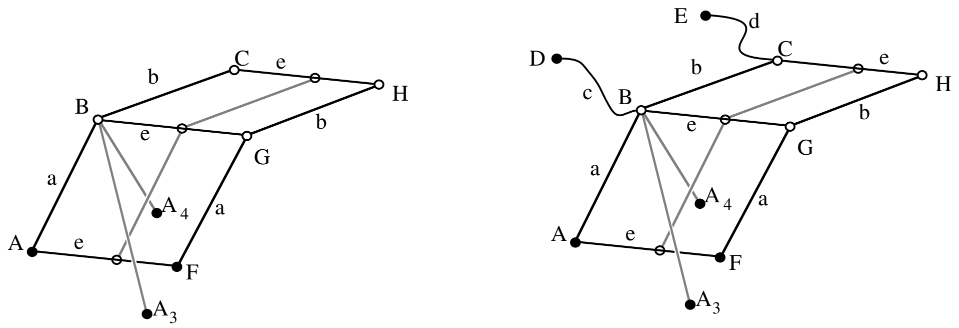

FIGURE 5. Translation Linkage

edge $\overline{BD}$ of length c, where D is fixed at some point z_2 (and A and A_i are fixed as before). By Lemma 3.6, we have

$$\begin{aligned}\mathcal{C}(\mathcal{L}') &= \rho_{\mathcal{L},B}^{-1}(\{|z-z_2|\leq c\}) \\ &= \{(u,v)\in S^1\times S^{n-1} \mid c\geq |z_1+au-z_2|\} = T\times S^{n-1}\end{aligned}$$

for some arc T of S^1, as long as we choose z_2 and c appropriately. For convenience, we choose $c=\sqrt{2}a$ and $z_2=z_1+aw_0$ for some $w_0\in S^1$. Then T will be the semicircle between $\pm w_0'$ which contains w_0, where w_0' is obtained by rotating w_0 by $\pi/2$. By Lemma 4.5, we know that $\rho_{\mathcal{L},C}$ restricts to an analytically trivial covering of $\{|z-z_1-aw_0'|<b\}$. But by checking the inverse image of a point, for example z_1+aw_0', we see that it is a one-fold cover, hence an analytic isomorphism. So now in $\mathcal{L}$, if we fix E at z_1+aw_0' and pick $d<b$, we see that $\rho_{\mathcal{L},C}:\mathcal{C}(\mathcal{L})\to\mathbb{R}^n$ is an analytic isomorphism to its image $|z-z_1-aw_0'|\leq d$.

4.6. **A Functional Linkage for Translation.** Now let us find a functional linkage for translation $z\mapsto z+z_0$ with restricted domain $|z|\leq r$. Consider the linkages $\mathcal{L}$ in Figure 5, which we will show to be functional for $z\mapsto z+z_0$ with restricted domain $|z|\leq r$. The right hand cabled linkage will be strongly functional.

Choose $a>2r$, $b\leq a$, and let $e=|z_0|$. We start with the sublinkage $\mathcal{L}'$ which is that of section 4.5, with z_1 to be determined later. We add a vertex F fixed at z_1+z_0, and add vertices G and H, and add edges $\overline{AF}$, $\overline{BG}$ and $\overline{CH}$ of length e, and add edges $\overline{FG}$ and $\overline{GH}$ of lengths a and b. We also rigidify the parallelograms $ABGF$ and $BCHG$.

We let C be the input vertex and H be the output vertex. Notice $\overline{AF}$, $\overline{BG}$, and $\overline{CH}$ are parallel, and so for any $\varphi\in\mathcal{C}(\mathcal{L})$ we must have $\varphi(H)=\varphi(C)+z_0$. So $\mathcal{L}$ is quasifunctional and we must only check that the restricted domain can be $|z|\leq r$.

We claim by Lemma 3.7 that $\rho_{\mathcal{L},\mathcal{L}'}:\mathcal{C}(\mathcal{L})\to\mathcal{C}(\mathcal{L}')$ is an isomorphism. This is because the positions of F, G, H, and the other three unnamed vertices used to rigidify the quadrilaterals are all polynomial functions of the positions of A, B, and C. Now the fact that $\rho_{\mathcal{L},C}$ doubly covers $|z|\leq r$ (for the left hand classical linkage) or singly covers $|z|\leq r$ (for the right hand cabled linkage) follows from the discussion of $\mathcal{C}(\mathcal{L}')$ in section 4.5, as long as we make appropriate choices of z_1, a, b, and w_0. For example, we may choose b so $r<b<a$, choose $w_0=e_1$, $w_0'=e_2$, and $z_1=-ae_2$.

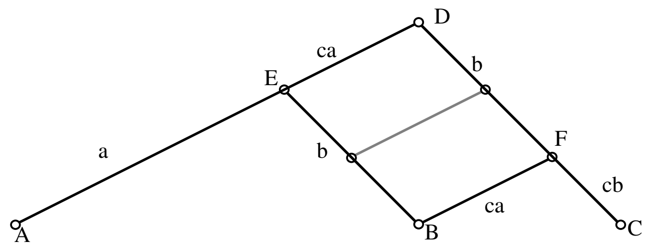

FIGURE 6. The Pantograph

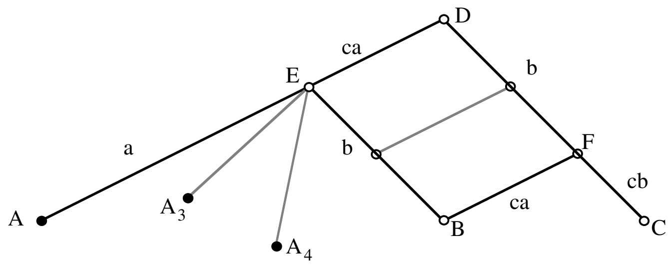

FIGURE 7. A functional linkage for scalar multiplication in $\mathbb{R}^4$

4.7. **A Functional Linkage for real scalar Multiplication.** Now let us find a functional linkage $\mathcal{L}$ for scalar multiplication $z \mapsto \lambda z$. To do this we marry the pantograph of Figure 6 with the linkage of section 4.5. To make the pantograph, we take a rigidified parallelogram $EDFB$ with side lengths b and ca. To this we add vertices A and C and sides $\overline{AE}$ of length a, $\overline{AD}$ of length $a + ca$, $\overline{FC}$ of length cb, and $\overline{CD}$ of length $b + cb$. Thus E and F are simulated interior joints.

For any realization φ of the pantograph

$$\varphi(C) - \varphi(A) = (1 + c)(\varphi(B) - \varphi(A)) \tag{4.2}$$

So the pantograph is a quasifunctional linkage by equation (4.2). For example if vertex A is fixed at 0 and B is the input and C is the output, it will be quasifunctional for $x \mapsto (1 + c)x$. But if $n > 2$, the configuration space is too big for it to be a functional linkage since any realization can be rotated about the line through A, B and C. Consequently, $\rho_{\mathcal{L},B}$ would not be finite to one, so we could not get a functional linkage. To take care of this problem, we will combine the pantograph with a linkage from section 4.5 to make it functional.

We divide the construction of a functional linkage for scalar multiplication into three cases, $\lambda > 1$, $0 < \lambda < 1$, and $\lambda < 0$. The remaining cases $\lambda = 0$ or $\lambda = 1$ are trivial functions which have trivial functional linkages.

If $\lambda > 1$ we take $c = \lambda - 1$, let B be the input vertex and let C be the output vertex, and fix A at 0. We add fixed vertices A_i for $n \geq i \geq 3$, fixed at ae_i, and edges $\overline{A_iE}$ of length $\sqrt{2}a$. So we have a section 4.5 sublinkage $\mathcal{L}'$ with vertices A, A_i, E, and B. Note that for any $\varphi \in \mathcal{C}(\mathcal{L})$ we have $\varphi(C) = \lambda\varphi(B)$. So $\mathcal{L}$ is quasifunctional. Figure 7 shows this linkage for $n = 4$ and $\lambda = 1 + c$.

By Lemma 3.7 we know that $\rho_{\mathcal{L},\mathcal{L}'} : \mathcal{C}(\mathcal{L}) \to \mathcal{C}(\mathcal{L}')$ is an isomorphism since the positions of D, F, and C are polynomial functions of A, E, and B. By the discussion of the linkage in section 4.5, we know that if a and b are chosen appropriately, then

$\rho_{\mathcal{L},B}$ double covers some disc $|z - z_0| \le r$. Hence $\mathcal{L}$ is functional with restricted domain $\{|z - z_0| \le r\}$.

To get a strong functional linkage, we add two fixed vertices and tether E and B to them with appropriate length cables as in section 4.5. By the discussion in section 4.5, we know that $\rho_{\mathcal{L},B}$ singly covers some disc $|z - z_0| \le r$. Consequently we get a strong functional linkage with restricted domain $\{|z - z_0| \le r\}$.

If $0 < \lambda < 1$, we take $c = 1/\lambda - 1$, let C be the input vertex, let B be the output vertex, and fix A at 0. We add fixed vertices A_i at $(a/\lambda)e_i$ and edges A_iD of length $\sqrt{2}a/\lambda$ for $3 \le i \le n$. By considering the sublinkage with vertices A, A_i, D, and C, we see as above that with appropriate choices of a and b, the linkage will be functional for $z \mapsto \lambda z$ with restricted domain $|z - z_0| \le r$. To get a strongly functional linkage, tether D and C appropriately to fixed vertices A_1 and A_2, as in section 4.5. For the next section it will be useful to point out how z_0 can be chosen. Looking back at the analysis of section 4.5, we see that we can if we wish pick z_0 to be any point in the circle of radius a/λ in the x_1x_2 plane.

If $\lambda < 0$, we take $c = -\lambda$, let A be the input vertex, C be the output vertex, and fix B at 0. We add fixed vertices B_i at be_i and edges B_iE of length $\sqrt{2}b$. Letting $\mathcal{L}'$ be the sublinkage with vertices B, B_i, E, and A, we see as above that $\mathcal{L}$ is functional for $z \mapsto \lambda z$ with restricted domain $|z - z_0| \le r$. To get a strongly functional linkage, tether E and A appropriately.

4.8. **A Functional Linkage for the Average.** Now let us find a functional linkage $\mathcal{L}$ for the average. Again $\mathcal{L}$ will be based on the pantograph of Figure 6. The input vertices will be A and C. The output vertex will be B. We let $c = 1$ and choose $a = b > r$. Note $\mathcal{L}$ is quasifunctional for $(z, w) \mapsto (z + w)/2$. However, this $\mathcal{L}$ could not be functional because D is free to rotate around the line through A, B, and C; hence $(\rho_{\mathcal{L},C}, \rho_{\mathcal{L},A})$ could not be finite to one. So we modify the pantograph as follows.

Let us start with the linkage $\mathcal{L}'''$ found in the previous section which is (strongly) functional for the map $z \mapsto z/2$ with restricted domain $U''' = \{|z - 2ae_1| \le 2r\}$. In particular, $\mathcal{L}'''$ is a pantograph together with a few more fixed vertices $A_\ell, \dots, A_n$, where $\ell = 1$ in the cabled linkage case and $\ell = 3$ in the classical linkage case. The fixed vertices are A fixed at 0, and A_i fixed at some z_i. The input vertex is C and the output is B. Let $\mathcal{L}''$ be obtained from $\mathcal{L}'''$ by unfixing all the fixed vertices of $\mathcal{L}'''$.

Using Lemma 4.2 and the functional linkage found above for translation, we may find a (strong) functional linkage $\mathcal{L}'$ for the function $z \mapsto (z + z_\ell, \dots, z + z_n)$ with restricted domain $U' = \{|z + ae_1| \le r\}$. Let the input vertex of $\mathcal{L}'$ be v and let the output vertices of $\mathcal{L}'$ be $w_\ell, \dots, w_n$. We form a linkage $\mathcal{L}$ by taking the disjoint union of $\mathcal{L}'$ and $\mathcal{L}''$, identifying v with A, and identifying w_i with A_i. Figure 8 shows the result for a classical linkage in $\mathbb{R}^3$.

By Lemma 3.1 we know that

$$\begin{aligned}\mathcal{C}(\mathcal{L}) &= \{(\varphi', \varphi'') \in \mathcal{C}(\mathcal{L}') \times \mathcal{C}(\mathcal{L}'') \mid \varphi'(v) = \varphi''(A) \text{ and } \varphi'(w_i) = \varphi''(A_i)\} \\ &= \{(\varphi', \varphi'') \in \mathcal{C}(\mathcal{L}') \times \mathcal{C}(\mathcal{L}'') \mid \varphi'(v) = \varphi''(A) \text{ and } \varphi'(v) + z_i = \varphi''(A_i)\}\end{aligned}$$

Thus we have an isomorphism

$$\beta \colon \mathcal{C}(\mathcal{L}') \times \mathcal{C}(\mathcal{L}''') \to \mathcal{C}(\mathcal{L})$$

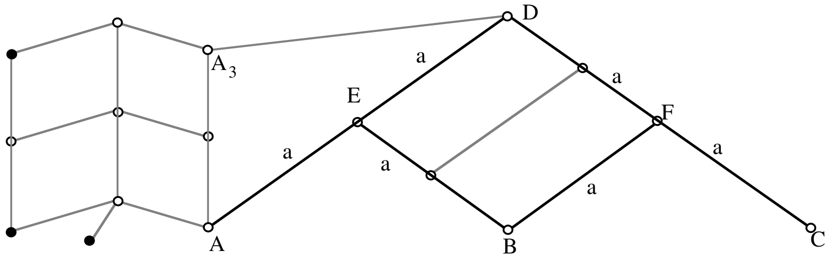

FIGURE 8. A functional linkage for the average in $\mathbb{R}^3$

given by $\beta(\varphi', \varphi''') = (\varphi', \varphi''' + \varphi'(v))$ where $\varphi''' + \varphi'(v)$ is the translate of φ''' by $\varphi'(v)$. Now

$$(\rho_{\mathcal{L},A}, \rho_{\mathcal{L},C}) \circ \beta(\varphi', \varphi''') = (\varphi'(v), \varphi'''(C) + \varphi'(v))$$

By (strong) functionality we have finite sets F' and F''' and analytic isomorphisms

$$\begin{aligned} \sigma' : U' \times F' \quad &\to \quad {\rho_{\mathcal{L}',v}}^{-1}(U') \\ \sigma''' : U''' \times F''' \quad &\to \quad {\rho_{\mathcal{L}''',C}}^{-1}(U''') \end{aligned}$$

so that $\rho_{\mathcal{L}',v}\sigma'(x, f) = x$ and $\rho_{\mathcal{L}''',C}\sigma'''(x, f) = x$, and so that F' and F''' are singletons in the cabled linkage case.

In the cabled linkage case, we also tether A to $-ae_1$ and tether C to ae_1 with cables of length r. This will insure that the domain of $\mathcal{L}$ is U given in (4.3) below.

So in any case, if

$$U = \{ (z, w) \in \mathbb{R}^n \times \mathbb{R}^n \mid r \geq |z - ae_1| \text{ and } r \geq |w + ae_1| \} \tag{4.3}$$

we have an analytic isomorphism

$$\sigma : U \times F' \times F''' \to (\rho_{\mathcal{L},A}, \rho_{\mathcal{L},C})^{-1}(U)$$

given by

$$\sigma(z, w, f', f''') = \beta(\sigma'(w, f'), \sigma'''(z - w, f'''))$$

In particular, $(\rho_{\mathcal{L},C}, \rho_{\mathcal{L},A})$ is an analytically trivial cover of U, and is an analytic isomorphism in the cabled linkage case. So $\mathcal{L}$ is (strongly) functional for the average.

As it happens, this is the only place where a cabled linkage is constructed with a flexible edge between two nonfixed vertices. If one wished, one could change the construction slightly to avoid this, by eliminating A_1 and A_2, but tethering D to some fixed vertex instead. One could then strengthen Theorem 2.1 to conclude that in addition, each flexible edge of $\mathcal{L}'$ is connected to a fixed vertex.

4.9. **A Functional Linkage for Inversion in a Line.** Let L be a line through 0, $z_0 \in L - 0$ and $K \subset L - 0$ compact. We will now construct a functional linkage $\mathcal{L}$ for the function $f : L \to L$ with restricted domain K, where $f(sz_0) = (1/s)z_0$. In the cabled linkage case we will make $\mathcal{L}$ strongly functional. Note that the input vertex v will be constrained to lie on the line L. Here we will restrict to the case $n \geq 3$. The $n = 2$ case requires a different construction, but was shown in [3] and [1] so we will not repeat it here.

It suffices to show this for L the x_1 axis, and $z_0 = e_1$, since any other line and z_0 may be obtained by rotation of $\mathcal{L}$, and rescaling all side lengths. Pick $0 < c < b$ to be determined later. Let $a = \sqrt{1 + b^2}$ and $d = \sqrt{1 + c^2}$.

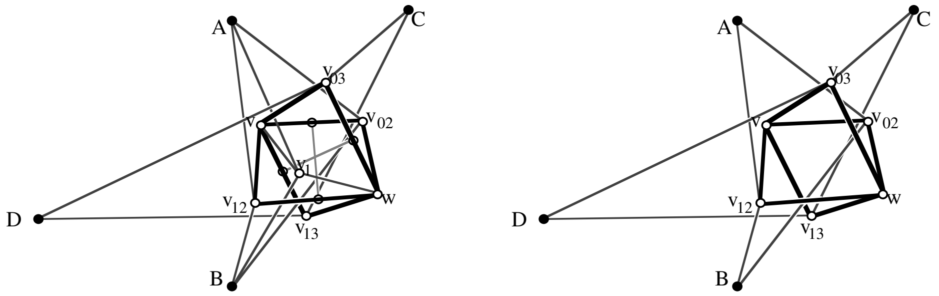

FIGURE 9. A functional linkage for inversion in a line in $\mathbb{R}^3$

For $i = 2, \dots, n$, let S_i be the circle

$$S_i = \{x \in \mathbb{R}^n \mid x_1^2 + x_i^2 = a^2, \text{ and } x_j = 0 \text{ for } j \neq 1, i\}$$

Using Lemma 4.6, find linkages $\mathcal{L}_{ij}$ with vertices v_{ij} so that $\mathcal{SC}(\mathcal{L}_{ij}, \{v_{ij}\}) = S_i$, and $\rho_{\mathcal{L}_{ij}, v_{ij}}$ is an isomorphism $i = 2, \dots, k$ and $j = 0, 1$. Form a linkage $\mathcal{L}$ by taking the disjoint union of the linkages $\mathcal{L}_{ij}$, adding vertices v and w, and putting in rigidified squares with vertices v, v_{i0}, w, v_{i1}, and side lengths b. In the cabled linkage case put in a cable between v and w with length $2c$. We also tether each v_{i0} to the point ae_i with a cable of length $\sqrt{2}a$. In the classical linkage case, use Lemma 4.6 to find a linkage $\mathcal{L}_1$ and vertex v_1 so that

$$\mathcal{SC}(\mathcal{L}_1, \{v_1\}) = \{x \in \mathbb{R}^n \mid x_1^2 + x_2^2 = d^2, \text{ and } x_j = 0 \text{ for } j \geq 3\}$$

and $\rho_{\mathcal{L}_1, v_1}$ is an isomorphism to this circle. Add $\mathcal{L}_1$ to $\mathcal{L}$ and put in edges $\overline{vv_1}$ and $\overline{wv_1}$ of length c.

Figure 9 shows a functional classical linkage for inversion in the x_1 axis of $\mathbb{R}^3$, viewed from a point (e, e, e). At the left is the full linkage which is somewhat complicated to decipher, the right hand linkage eliminates the rigidifying edges and the vertex v_1 to give the essentials. Vertices A and B are fixed at points on the x_3-axis and C and D are fixed at points on the x_2-axis. Thus v_{02}, v_{12}, and v_1 are restricted to circles in the x_1x_2 plane, and v_{03} and v_{13} are restricted to a circle in the x_1x_3 plane. The thick edges form the important part of the linkage, two rigidified squares.

Let us show that $\mathcal{L}$ is quasifunctional for inversion. Pick any $\varphi \in \mathcal{C}(\mathcal{L})$ and let $\varphi(v) = x$, $\varphi(w) = y$ and $\varphi(v_{ij}) = x_{ij}$. If $x_{i0} = x_{i1}$ then $|x - y| = 2b > 2c$. So in the cabled linkage case the cable between v and w guarantees that $x_{i0} \neq x_{i1}$. In the classical case, $|x - y| \leq |x - \varphi(v_1)| + |\varphi(v_1) - y| = 2c$, so again $x_{i0} \neq x_{i1}$.

Because we rigidified the quadrilateral $xx_{i0}yx_{i1}$, we know that $(x + y)/2 = (x_{i0} + x_{i1})/2$ since these are both midpoints of the intersecting diagonals. So $x + y$ is in the x_1x_i plane for all $i \geq 2$. Since $n \geq 3$, we then conclude that $x + y$ is on the x_1 axis. So $(x + y)/2 = \alpha e_1$ for some α. Since αe_1 is at the midpoint of a chord of the circle S_i, we must have $x_{ij} = \alpha e_1 + \epsilon_i(-1)^j \beta e_i$ where $\beta = \sqrt{a^2 - \alpha^2} > 0$ and $\epsilon_i = \pm 1$. Note that in the cabled linkage case that the cable tethering v_{i0} to ae_i means that

$$2a^2 \geq |\alpha e_1 + \epsilon_i \beta e_i - ae_i|^2 = \alpha^2 + \beta^2 - 2\epsilon_i \beta a + a^2 = 2a^2 - 2\epsilon_i \beta a \tag{4.4}$$

so $\epsilon_i \beta a \geq 0$ and hence $\epsilon_i = 1$.

Now

$$b^2 = |x - x_{ij}|^2 = |x|^2 + |x_{ij}|^2 - 2x \cdot x_{ij} = |x|^2 + a^2 - 2x \cdot x_{ij} \tag{4.5}$$

Consequently $x \cdot (x_{i0} - x_{i1}) = 0$ so $x \cdot e_i = 0$ for all $i \geq 2$. So $x = \gamma e_1$ for some γ. Note from equation (4.5) that $b^2 = \gamma^2 + a^2 - 2\gamma\alpha$, so $\gamma^2 - 2\gamma\alpha + 1 = 0$. Hence

$$\begin{aligned} \gamma &= \alpha \pm \sqrt{\alpha^2 - 1} \\ y &= 2(x+y)/2 - x = (2\alpha - \gamma)e_1 = (1/\gamma)e_1 \end{aligned}$$

So we see that $\mathcal{L}$ is quasifunctional for $se_1 \mapsto (1/s)e_1$.

Let us now see what the restricted domain can be. Since $|x - y| \leq 2c$ we must have $|\gamma - 1/\gamma| \leq 2c$ so solving we find we must have $\gamma \in A$ where

$$A = \{t \in \mathbb{R} \mid d - c \leq |t| \leq d + c\}$$

By choosing c large enough, we may ensure that $K \subset Ae_1$.

We will show that the image of $\rho_{\mathcal{L},v}$ is Ae_1. Moreover in the cabled linkage case $\rho_{\mathcal{L},v}$ is an analytic isomorphism to Ae_1, and in the classical case $\rho_{\mathcal{L},v}$ restricts to an analytically trivial cover of $A'e_1$ where A' is the interior of A.

So we need to solve for the positions of the vertices in terms of γ. Pick any $\gamma \in A$. We have already seen that if $\varphi \in \mathcal{C}(\mathcal{L})$ and $\varphi(v) = \gamma e_1$, then $\varphi(w) = (1/\gamma)e_1$. Moreover, if $\alpha = (\gamma + 1/\gamma)/2$ and $\beta = \sqrt{a^2 - \alpha^2}$ then

$$\varphi(v_{ij}) = \alpha e_1 + \epsilon_i(-1)^j \beta e_i \tag{4.6}$$

where $\epsilon_i = \pm 1$, and $\epsilon_i = 1$ in the cabled linkage case. In the cabled linkage case, the cable between v and w means we must have $|\gamma - 1/\gamma| \leq 2c$ which is true for all $\gamma \in A$. Also the cables between ae_i and v_{i0} require that $|\varphi(v_{i0}) - ae_i| \leq \sqrt{2}a$ which follows from (4.4) since $\epsilon_i = 1$.

So in the cabled linkage case we have seen that $\rho_{\mathcal{L},v}$ is an analytic isomorphism from $\mathcal{C}(\mathcal{L})$ to Ae_1, hence $\mathcal{L}$ is strongly functional for f with domain $Ae_1 \supset K$.

In the classical case, it only remains to solve for $\varphi(v_1)$. We know that $\varphi(v_1)$ must be in the circle of radius d in the x_1x_2 plane which means that $\varphi(v_1) = se_1 + te_2$ for some s, t with $s^2 + t^2 = d^2$. So we must have

$$\begin{aligned} 0 = |\varphi(v_1) - \varphi(v)|^2 - c^2 &= (s - \gamma)^2 + t^2 - c^2 \\ &= d^2 - c^2 - 2s\gamma + \gamma^2 = 1 - 2s\gamma + \gamma^2 \\ 0 = |\varphi(v_1) - \varphi(w)|^2 - c^2 &= (s - 1/\gamma)^2 + t^2 - c^2 \\ &= d^2 - c^2 - 2s/\gamma + 1/\gamma^2 = 1 - 2s/\gamma + 1/\gamma^2 \end{aligned}$$

From either of these equations we then solve for s and obtain $s = \alpha$. Consequently,

$$\varphi(v_1) = \alpha e_1 + \epsilon_1 \sqrt{d^2 - \alpha^2}\, e_2 \tag{4.7}$$

where $\epsilon_1 = \pm 1$.

So in the end we see we have a map $\sigma\colon A \times \{-1,1\}^n \to \mathcal{C}(\mathcal{L})$ where the $\{-1,1\}^n$ chooses the signs ϵ_i. Moreover, looking at equations (4.7) and (4.6) and recalling that $a > d$, we see that σ restricts to an analytic isomorphism wherever we have $0 < d^2 - \alpha^2 = c^2 - (\gamma - 1/\gamma)^2$, i.e., on $A' \times \{-1,1\}^n$. So $\mathcal{L}$ is functional for f with restricted domain $A'e_1$.

This construction of inversion in a line is the one place in this paper where we have used $n \geq 3$. If $n = 2$, a different construction is needed, see [1] or [3]. The construction above with $n = 2$ would actually give you inversion through the circle, since the domain would not be restricted to a single line.

4.10. **A Functional Linkage for Orthogonal Projection to a Line.** Let $g\colon \mathbb{R}^n \to \mathbb{R}^n$ be orthogonal projection to a line L. Let $K \subset \mathbb{R}^n$ be compact. We will now construct a functional linkage $\mathcal{L}$ for g with restricted domain K. As a bonus, we will at the same time construct a functional linkage for reflection $f\colon \mathbb{R}^n \to \mathbb{R}^n$ about the line L, although we do not use this fact.

Before finding this linkage, we prove the following Lemma:

Lemma 4.7. *Given any compact line segment $T \subset \mathbb{R}^n$, there is a linkage $\mathcal{L}$ and a $v \in \mathcal{V}(\mathcal{L})$ so that $T = \mathcal{SC}(\mathcal{L}, \{v\})$. If we insist that $\mathcal{L}$ be a classical linkage then we may ensure that $\rho_{\mathcal{L},v}$ restricts to an analytically trivial cover of the interior T' of T. If $\mathcal{L}$ is allowed to be a cabled linkage, we may make $\rho_{\mathcal{L},v}\colon \mathcal{C}(\mathcal{L}) \to T$ be an analytic isomorphism.*

Proof. It suffices to prove this for only one line segment T, since any other line segment may be obtained from T by translation, rotation, and rescaling. Let $\mathcal{L}_0$ be the (strong) functional linkage constructed in section 4.9 for inversion in the x_1 axis. Take $c = 3/4$ and $d = 5/4$ so the domain is $U_0 = \{se_1 \mid 1/2 \leq |s| \leq 2\}$. Let $\mathcal{L}_1$ be obtained from $\mathcal{L}_0$ by translating by $2e_1$, so the domain of $\mathcal{L}_1$ is $U_1 = \{se_1 \mid 1/2 \leq |s-2| \leq 2\}$. Let v_i be the input vertex of $\mathcal{L}_i$. Let $\mathcal{L}$ be obtained from the disjoint union of $\mathcal{L}_0$ and $\mathcal{L}_1$ by gluing the inputs v_0 and v_1 together. Then $\mathcal{C}(\mathcal{L})$ is the fiber product of the maps $\rho_{\mathcal{L}_0,v_0}$ and $\rho_{\mathcal{L}_1,v_1}$. Consequently, if $v = v_0 = v_1$ is the glued vertex, we have

$$\rho_{\mathcal{L},v}(\mathcal{C}(\mathcal{L})) = U_0 \cap U_1 = \{se_1 \mid 1/2 \leq s \leq 3/2\} = T$$

Let U_i' and T' be the interiors of U_i and T in the x_1 axis. We have a finite set F and analytic isomorphisms $\sigma_i\colon U_i' \times F \to \rho^{-1}_{\mathcal{L}_i,v_i}(U_i')$ so that $\rho_{\mathcal{L}_i,v_i}(\sigma_i(u,f)) = u$. Looking at $\mathcal{C}(\mathcal{L})$ as a fiber product

$$\mathcal{C}(\mathcal{L}) = \{(\varphi_0, \varphi_1) \in \mathcal{C}(\mathcal{L}_0) \times \mathcal{C}(\mathcal{L}_1) \mid \varphi_0(v_0) = \varphi_1(v_1)\}$$

we have an analytic isomorphism $\sigma\colon T' \times F \times F \to \rho^{-1}_{\mathcal{L},v}(T')$ given by $\sigma(u, f_0, f_1) = (\sigma_0(u, f_0), \sigma_1(u, f_1))$.

In the cabled linkage case, we may take each $\rho_{\mathcal{L}_i,v_i}$ to be an analytic isomorphism and hence $\rho_{\mathcal{L},v}\colon \mathcal{C}(\mathcal{L}) \to T$ is an analytic isomorphism also. ☐

We now proceed to find a functional linkage for projection g to a line L. After translation and rotation, we may as well assume that L is the x_1 axis, and so $g(x) = x_1 e_1$. Choose r so that $K \subset \{|z| < r/2\}$. Let $U \subset \mathbb{R}^n$ be the set of points of distance $\leq r$ from L. Define $\psi_i : U \to L$, $i = 0, 1$ by

$$\psi_i(x) = g(x) + (-1)^i \sqrt{r^2 - |x - g(x)|^2}\, e_1$$

Note that $\psi_i(x)$ are the two points on L with distance r from x. For $x \in K$ we must have $|g(x)| \leq |x| < r/2$ and $\sqrt{r^2 - |x-g(x)|^2} > \sqrt{r^2 - (r/2)^2} = \sqrt{3}r/2$. So if $L_0 = \{te_1 \mid t > 0\}$ and $L_1 = \{te_1 \mid t < 0\}$ we must have $\psi_i(K) \subset L_i$.

Choose closed line segments T_i in L_i so that $\psi_i(K)$ is contained in the interior T_i' of T_i, $i = 0, 1$. By Lemma 4.7, we may choose linkages $\mathcal{L}_0$ and $\mathcal{L}_1$ with vertices v_i so that $\mathcal{SC}(\mathcal{L}_i, \{v_i\}) = T_i$, and $\rho_{\mathcal{L}_i,v_i}\colon \mathcal{C}(\mathcal{L}_i) \to L$ is an analytically trivial covering of T_i', and in fact in the cabled linkage case it is an analytic isomorphism to T_i.

Form $\mathcal{L}$ by taking the disjoint union of $\mathcal{L}_0$ and $\mathcal{L}_1$, adding two vertices v and w, and forming a rigidified square $v_1 v v_0 w$ with side length r. To get a functional linkage for g, we will be a bit specific about this rigidification. We will add three vertices, v_2, v_3, v_4. We will place v_4 at the midpoint of the edge $\overline{v_1 v}$, place v_3 at

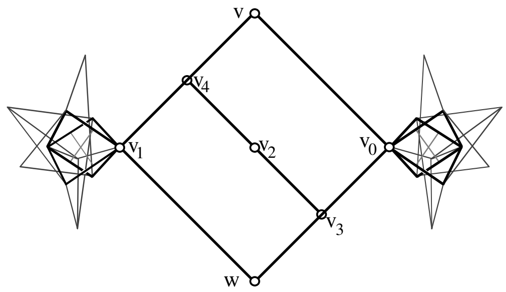

FIGURE 10. A functional linkage for projection to a line in $\mathbb{R}^3$

the midpoint of the edge $\overline{v_0 w}$, and place v_2 at the midpoint of the edge $\overline{v_3 v_4}$. Thus, in any realization, v_2 will be at the exact center of the parallelogram $v_1 v v_0 w$.

For a functional linkage for projection g, set v to be the input vertex and v_2 to be the output vertex. For a functional linkage for reflection f, set v to be the input vertex and w to be the output vertex.

Pick any $\varphi \in \mathcal{C}(\mathcal{L})$ and let $x = \varphi(v)$. Since $\varphi|_{\mathcal{V}(\mathcal{L}_i)} \in \mathcal{C}(\mathcal{L}_i)$, we know that $\varphi(v_i) \in T_i \subset L_i$. Since T_0 and T_1 are disjoint, we know that $\varphi(v_0) \neq \varphi(v_1)$. Consequently $\varphi(v_i) = \psi_i(x)$. Note that

$$\begin{aligned}\varphi(w) &= \varphi(v_0) + \varphi(v_1) - \varphi(v) = \psi_0(x) + \psi_1(x) - x = 2g(x) - x = f(x)\\ \varphi(v_2) &= (\varphi(v) + \varphi(w))/2 = (x + 2g(x) - x)/2 = g(x)\end{aligned}$$

so $\mathcal{L}$ is quasifunctional for f and g.

Let $\sigma_i \colon T_i' \times F_i \to \rho^{-1}_{\mathcal{L}_i, v_i}(T_i')$ be analytic isomorphisms with $\rho_{\mathcal{L}_i, v_i}(\sigma_i(z, f_i)) = z$. Then we get an analytic isomorphism $\sigma \colon K \times F_0 \times F_1 \to \rho^{-1}_{\mathcal{L}, v}(K)$ given by

$$\begin{aligned}\sigma(x, f_0, f_1)|_{\mathcal{L}_i} &= \sigma_i(\psi_i(x), f_i)\\ \sigma(x, f_0, f_1)(v) &= x\\ \sigma(x, f_0, f_1)(w) &= f(x)\\ \sigma(x, f_0, f_1)(v_2) &= g(x)\\ \sigma(x, f_0, f_1)(v_3) &= (f(x) + \psi_0(x))/2\\ \sigma(x, f_0, f_1)(v_4) &= (x + \psi_1(x))/2\end{aligned}$$

So $\mathcal{L}$ is functional for f and g with restricted domain K, and in the cabled linkage case it is strongly functional since the F_i have just one point each and σ may be extended to an analytic isomorphism $\sigma \colon (\psi_0^{-1}(T_0) \cap \psi_1^{-1}(T_1)) \times F_0 \times F_1 \to \mathcal{C}(\mathcal{L})$.

The astute reader will notice that the linkage shown in Figure 10 is simpler than that constructed in the text (which would be hopelessly cluttered with two copies of the linkage from Figure 9 on each side). The linkage shown will still work, the only difference is that each T_i will be a double interval.

5. PROOFS OF THEOREMS

Now that Theorem 2.1 is proven, we can prove the theorems stated in the first section. First we prove a special case of Theorem 1.3.

Proposition 5.1. *Suppose $Z \subset (\mathbb{R}^n)^k$ is compact, $n \geq 2$. The following are equivalent:*

1. *There is a classical linkage $\mathcal{L}$ and a $W \subset \mathcal{V}(\mathcal{L})$ so that $\mathcal{SC}(\mathcal{L}, W) = Z$.*
2. *Z is a semialgebraic set.*

Proof. The implication 1 implies 2 follows from the Tarski-Seidenberg Theorem, [6], because $\mathcal{C}(\mathcal{L})$ is an algebraic set and $\mathcal{SC}(\mathcal{L}, W)$ is its image under projection.

Now let us show that 2 implies 1. By Lemma 3.1 of [2] that there is a polynomial map $q\colon Y \to (\mathbb{R}^n)^k$ from some compact real algebraic set Y so that $Z = q(Y)$. By taking the graph of q, we may as well assume that $Y \subset (\mathbb{R}^n)^{k+m}$ and q is projection to the first nk coordinates. Pick a polynomial $p\colon (\mathbb{R}^n)^{k+m} \to \mathbb{R}$ so that $Y = p^{-1}(0)$. By Theorem 2.1, there is a functional classical linkage $\mathcal{L}'$ for the map $x \mapsto p(x)e_1$, with restricted domain Y and with distinct input vertices. Let $\mathcal{L}$ be obtained from $\mathcal{L}'$ by fixing the output vertex at 0. Let W be the first k input vertices of $\mathcal{L}'$ and let U be all the input vertices of $\mathcal{L}'$. Then by Lemma 3.5, $\mathcal{SC}(\mathcal{L}, U) = Y$, so $\mathcal{SC}(\mathcal{L}, W) = Z$. □

We are now able to prove Theorems 1.4 and 1.5.

Proof. (of Theorem 1.4) Note that part 3 implies part 2 since any m points of $\mathbb{R}^n$ are contained in some affine subspace with dimension $m-1$. Note also that $\mathcal{C}(\mathcal{L}) = \mathcal{SC}(\mathcal{L}, \mathcal{V}(\mathcal{L}))$ so it suffices to prove the $\mathcal{SC}(\mathcal{L}, W)$ results only.

To see part 3, let $b = \dim T$. We may pick $\beta \in \mathrm{Euc}(n)$ so that $\beta(T) = \mathbb{R}^b \times 0 \subset \mathbb{R}^b \times \mathbb{R}^{n-b}$. Then if $\gamma \in O(n-b)$ we have $\beta^{-1}\gamma\beta\mathcal{L} = \mathcal{L}$. So $\mathcal{SC}(\mathcal{L}, W)$ is invariant under the conjugate of $O(n-b)$ by β. We know $\mathcal{C}(\mathcal{L})$ is a quasialgebraic set, so it is closed. But it is also contained in a ball of radius d around the image of a fixed vertex, where d is the sum of the lengths of all edges of $\mathcal{L}$. (We are using connectedness here.) So we see that $\mathcal{C}(\mathcal{L})$ is compact. So then $\mathcal{SC}(\mathcal{L}, W)$ is compact, since it is the image of the compact $\mathcal{C}(\mathcal{L})$ under projection.

To see part 1, note that by Lemma 3.3, $\mathcal{SC}(\mathcal{L}, W)$ is invariant under the action of $\mathrm{Euc}(n)$. If W is empty, then $\mathcal{SC}(\mathcal{L}, W)$ is a single point which is compact, so we may assume that W has $k > 0$ vertices. Recall $\mathrm{Tran}(n) \subset \mathrm{Euc}(n)$ is the subgroup of translations. We may identify the quotient $\mathcal{SC}(\mathcal{L}, W)/\mathrm{Tran}(n)$ with

$$Z_0 = \{(z_1, \dots, z_k) \in \mathcal{SC}(\mathcal{L}, W) \mid z_k = 0\}$$

This is compact by part 2, since it is $\mathcal{SC}(\mathcal{L}', W)$ for the linkage $\mathcal{L}'$ formed from $\mathcal{L}$ by fixing the k-th vertex in W to 0. Consequently, the quotient $\mathcal{SC}(\mathcal{L}, W)/\mathrm{Euc}(n) = Z_0/O(n)$ is compact. □

Proof. (of Theorem 1.5) If Z is empty, we may easily prove this by choosing any $\mathcal{L}$ which includes a triangle which violates the triangular inequality. So we may assume Z is nonempty.

Let us first prove parts 2 and 3. First note that in part 2, by replacing Z by $\beta(Z)$ for some $\beta \in \mathrm{Euc}(n)$, we may assume that $G = O(m)$, acting on the last m coordinates of $\mathbb{R}^n$. By Proposition 5.1, we may find a linkage $\mathcal{L}'$ and a $W \subset \mathcal{V}(\mathcal{L}')$ so that $\mathcal{SC}(\mathcal{L}', W) = Z$. Throw away all connected components of $\mathcal{L}'$ which do not contain any vertices in W or any fixed vertices. By Lemma 3.2, doing so does not change $\mathcal{SC}(\mathcal{L}', W)$. By adding some isolated fixed vertices to $\mathcal{L}'$ if necessary, we may assume that there is a vertex fixed at 0, a vertex fixed at each e_i, $i = 1, \dots, n$ and a vertex fixed at $\sum_{i=1}^n e_i$. Adding an isolated fixed vertex to $\mathcal{L}'$ does not change $\mathcal{SC}(\mathcal{L}', W)$.

Let the fixed vertices of $\mathcal{L}'$ be $\{v_0, \dots, v_b\}$ where v_i is fixed to the point z_i. We may suppose $z_0 = 0$, $z_i = e_i$, $i = 1, \dots n$, and $z_{n+1} = \sum_{i=1}^n e_i$. For each pair i, j

with $z_i \neq z_j$ put in an edge $v_i v_j$ of length $|z_i - z_j|$, if it is not already there. This will not change $\mathcal{SC}(\mathcal{L}', W)$. Note we did not attempt to add any zero length edges, which would not be allowed.

Let $\mathcal{L}''$ be obtained from $\mathcal{L}'$ by only fixing the vertices v_i for $i \leq n$. We claim that $\mathcal{SC}(\mathcal{L}', W) = \mathcal{SC}(\mathcal{L}'', W)$. One inclusion $\mathcal{SC}(\mathcal{L}', W) \subset \mathcal{SC}(\mathcal{L}'', W)$ is trivial. So let us see the other inclusion. Pick any $\varphi \in \mathcal{C}(\mathcal{L}'')$. We claim that in fact $\varphi(v_i) = z_i$ for all i. To see this, note first that two different points in $\mathbb{R}^n$ can not have the same distances from $n+1$ points in general position, see Lemma 7.1. Here you can interpret general position to mean that their convex hull has nonempty interior. Consequently $\varphi'(v_{n+1}) = z_{n+1}$ since the $n+1$ edges $\overline{v_i v_{n+1}}$, $i \leq n$ have lengths $|z_{n+1} - z_i|$, so

$$|\varphi'(v_{n+1}) - z_i| = |\varphi'(v_{n+1}) - \varphi'(v_i)| = |z_{n+1} - z_i|$$

For any $j > n+1$, there are edges in $\mathcal{L}$ from v_j to at least $n+1$ of the v_i, $i \leq n+1$, and consequently $\varphi'(v_j) = z_j$ since any $n+1$ of the z_i, $i \leq n+1$ are in general position. Consequently, $\varphi \in \mathcal{C}(\mathcal{L}')$. So $\varphi|_W \in \mathcal{SC}(\mathcal{L}', W)$, and we have shown that $\mathcal{SC}(\mathcal{L}', W) = \mathcal{SC}(\mathcal{L}'', W)$.

We claim that $\mathcal{L}''$ is connected. Note Z is compact since its quotient by $O(m)$ is compact. By Lemma 3.9, we see that each connected component of $\mathcal{L}''$ has at least one fixed vertex. But the $n+1$ fixed vertices are connected to each other, hence there is only one connected component, so part 3 is proven.

Let us now prove part 2. Let $\mathcal{L}$ be obtained from $\mathcal{L}''$ by only fixing the vertices v_i for $i \leq n-m$, and not fixing any of the other vertices of $\mathcal{L}'$. We claim that $\mathcal{SC}(\mathcal{L}'', W) = \mathcal{SC}(\mathcal{L}, W)$. Again, one inclusion $\mathcal{SC}(\mathcal{L}'', W) \subset \mathcal{SC}(\mathcal{L}, W)$ is trivial. So let us see the other inclusion. Pick any $\varphi \in \mathcal{C}(\mathcal{L})$. Now if $1 \leq i \leq n$ and $1 \leq j \leq n$, $j \neq i$ we have $|\varphi(v_i)| = |z_i| = 1$, and $|\varphi(v_i) - \varphi(v_j)| = |z_i - z_j| = \sqrt{2}$. Consequently, the vectors $\varphi(v_1), \ldots, \varphi(v_n)$ form an orthonormal set. Since $\varphi(v_i) = e_i$ for $i \leq n-m$, we may choose a $\beta \in O(m)$, acting on the last m coordinates of $\mathbb{R}^n$, so that $\beta\varphi(v_i) = z_i$ for all $i \leq n$. For convenience, let $\varphi' = \beta \circ \varphi$. Note that $\varphi' \in \mathcal{C}(\mathcal{L}'')$. So $\varphi'|_W \in \mathcal{SC}(\mathcal{L}'', W) = Z$. By $O(m)$ invariance of Z, we know that $\beta^{-1} \circ \varphi'|_W \in Z$ also. But $\beta^{-1} \circ \varphi'|_W = \varphi|_W$, so $\varphi|_W \in Z$. So we have shown that $\mathcal{SC}(\mathcal{L}, W) \subset \mathcal{SC}(\mathcal{L}'', W)$, and hence $\mathcal{SC}(\mathcal{L}, W) = Z$. So part 2 is proven.

Now let us prove part 1. Let $\mathrm{Tran}(n) \subset \mathrm{Euc}(n)$ be the subgroup of translations. Then we may identify the quotient $Z/\mathrm{Tran}(n)$ with

$$Z_0 = \{(z_1, \ldots, z_k) \in Z \mid z_k = 0\}$$

Note Z_0 is invariant under the diagonal action of $O(n)$ and $Z_0/O(n) = Z/\mathrm{Euc}(n)$. So $Z_0/O(n)$ is compact and hence Z_0 is compact. Also Z_0 is semialgebraic since it is the intersection of semialgebraic sets. Define $Z_0' \subset (\mathbb{R}^n)^{k-1}$ by $Z_0 = Z_0' \times 0$. Note Z_0' is compact and semialgebraic since it is a projection of Z_0. Also Z_0' is $O(n)$ invariant since Z_0 is. So by part 2, there is a linkage $\mathcal{L}'$ and a $W' \subset \mathcal{V}(\mathcal{L}')$ so that $\mathcal{SC}(\mathcal{L}', W') = Z_0'$. Furthermore, there is only one fixed vertex v_0 of $\mathcal{L}'$ and that vertex is fixed at the point 0.

Consider the linkage $\mathcal{L}$ formed from $\mathcal{L}'$ by unfixing the vertex v_0. So $\mathcal{L}$ has no fixed vertices. Let $W = \{w_1, \ldots, w_k\}$ be such that $W' = \{w_1, \ldots, w_{k-1}\}$ and $w_k = v_0$. Then by Theorem 1.4, $\mathcal{SC}(\mathcal{L}, W)$ is invariant under the action of $\mathrm{Euc}(n)$, as is Z. But then $\mathcal{SC}(\mathcal{L}, W) = Z$ since they both have the same intersection with $(\mathbb{R}^n)^{k-1} \times 0$, (namely Z_0), and so are both the union of $\mathrm{Tran}(n)$ orbits of the same set. $\square$

We may now prove Theorem 1.3.

Proof. It is immediate that condition 2 implies condition 1. Now let us see why condition 1 implies condition 3. By Lemma 3.2 it suffices to consider the case where $\mathcal{L}$ is connected, but this case follows from Theorem 1.4. To see condition 3 implies condition 2, note that by Lemma 3.2 it suffices to consider the case where X is itself virtually compact. But then this case follows from Theorem 1.5. $\square$

Now let us prove Theorems 1.2 and 1.1 together.

Proof. Suppose $\mathcal{L}$ is a linkage. We need to show that $\mathcal{C}(\mathcal{L})$ is isomorphic to $X \times (\mathbb{R}^n)^k$ for some compact quasialgebraic set X, and X is algebraic if $\mathcal{L}$ is classical. By Lemma 3.2, we may suppose that $\mathcal{L}$ is connected. If $\mathcal{L}$ has any fixed vertices, then $\mathcal{C}(\mathcal{L})$ is compact by Theorem 1.4. So we have reduced to the case where $\mathcal{L}$ has no fixed vertices. Form a linkage $\mathcal{L}'$ from $\mathcal{L}$ by fixing one of its vertices v to 0. By Theorem 1.4, $\mathcal{C}(\mathcal{L}')$ is compact. By Lemma 3.4, $\mathcal{C}(\mathcal{L})$ is isomorphic to $\mathcal{C}(\mathcal{L}') \times \mathbb{R}^n$. So the first parts of the two theorems are proven.

Now let us prove the second part of Theorem 1.2. Let $X \subset (\mathbb{R}^n)^m$ be a compact quasialgebraic set and $k \geq 0$. Find polynomials $r_i \colon (\mathbb{R}^n)^m \to \mathbb{R}$, $i = 1, \ldots, \ell$ and $\ell' \leq \ell$ so that

$$(5.1) \quad X \quad = \quad \{\, x \in (\mathbb{R}^n)^m \mid r_i(x) = 0 \text{ for } i \leq \ell' \text{ and } r_i \geq 0 \text{ for } \ell' < i \leq \ell \,\}$$

Let $r \colon (\mathbb{R}^n)^m \to (\mathbb{R}^n)^\ell$ be the map $(r_1 e_1, r_2 e_1, \ldots, r_\ell e_1)$. Let $\mathcal{L}'$ be a strong functional linkage for the map r with restricted domain X. Let its input and output maps be q and p respectively. Let $v_1, \ldots, v_\ell$ be the output vertices. By compactness we may pick a $d > 0$ so that $r_i(x) \leq 2d$ for all $x \in X$, $i = 1, \ldots, \ell$.

Form a linkage $\mathcal{L}$ as follows. Take $\mathcal{L}'$ and add a vertex u_i for each $\ell' < i \leq \ell$. For each $i \leq \ell'$ fix the vertex v_i to 0. For each $\ell' < i \leq \ell$ attach a flexible edge of length d between v_i and u_i, and fix u_i to de_1.

By Lemmas 3.5 and 3.6, $\mathcal{C}(\mathcal{L})$ is naturally isomorphic to

$$Y = \{\varphi \in \mathcal{C}(\mathcal{L}') \mid \varphi(v_i) = 0 \text{ for } i \leq \ell' \text{ and } |\varphi(v_i) - de_1| \leq d \text{ for } \ell' < i \leq \ell\}$$

but $p(\varphi) = (\varphi(v_1), \ldots, \varphi(v_\ell))$ and $p = r \circ q$, so by (5.1), $Y = q^{-1}(X)$. But q is an analytic isomorphism, so $\mathcal{C}(\mathcal{L})$ is analytically isomorphic to X. Since this isomorphism $q \circ \rho_{\mathcal{L},\mathcal{L}'}$ is just given by coordinate projection, we immediately obtain the map α.

So we have done the compact case, $k = 0$. In general if $k > 0$, just add k isolated unfixed vertices to $\mathcal{L}$. By Lemma 3.2 the resulting linkage will have configuration space $\mathcal{C}(\mathcal{L}) \times (\mathbb{R}^n)^k$.

The second part of Theorem 1.1 is proven similarly. We start with a compact algebraic set X. There are no inequalities, so $\ell' = \ell$. However, since we want the linkage $\mathcal{L}'$ to be classical, it may not be strongly functional, just functional. So we have an analytic isomorphism $\sigma \colon X \times F \to q^{-1}(X)$ for some finite set F. We form $\mathcal{L}$ by fixing all the output vertices of $\mathcal{L}'$ to 0. Then by Lemma 3.5 we know that

$$\mathcal{C}(\mathcal{L}) = p^{-1}(0) = q^{-1} r^{-1}(0) = q^{-1}(X)$$

and the result follows. $\square$

Proof. (of Theorem 1.6) As in the proof of Theorem 1.5, we may assume that Z is nonempty.

Let us prove part 3. By Theorem 1.1 there is a classical linkage $\mathcal{L}'$, a finite set F, and an analytic $\alpha\colon Z \times F \to (\mathbb{R}^n)^m$ so that the map $(x, f) \mapsto (x, \alpha(x, f))$ is an analytic isomorphism from $Z \times F$ to $\mathcal{C}(\mathcal{L}')$. Note that each component of $\mathcal{L}'$ must have a fixed vertex by Lemma 3.9 with $W = \mathcal{V}(\mathcal{L})$. We now proceed as in the proof of Theorem 1.5. By adding some isolated fixed vertices to $\mathcal{L}'$ if necessary, we may assume that there is a vertex v_0 fixed at 0, a vertex v_i fixed at each e_i, $i = 1, \ldots, n$ and a vertex fixed at $\sum_{i=1}^{n} e_i$. Add edges of the appropriate length between the fixed vertices. Finally, unfix all fixed vertices except $v_0, v_1, \ldots, v_n$. Just as in the proof of Theorem 1.5, the resulting classical linkage $\mathcal{L}$ has $\mathcal{C}(\mathcal{L}) = \mathcal{C}(\mathcal{L}')$.

Let us now prove part 1. After rotation and translation, we may assume $G = O(1)$. So G has just one nontrivial element τ, reflection about the hyperplane $\{x_n = 0\}$. By part 3, there is a classical linkage $\mathcal{L}'$ with $n + 1$ fixed vertices $v_0, \ldots, v_n$ fixed at 0 and e_i so that $\mathcal{C}(\mathcal{L}')$ is analytically isomorphic to $Z \times F$ for some finite F. Let $\mathcal{L}$ be obtained from $\mathcal{L}'$ by unfixing the vertex v_n, but adding edges $\overline{v_n v_0}$ of length 1 and $\overline{v_n v_i}$ of length $\sqrt{2}$ for $1 \le i \le n-1$, if these are not already present. Pick any $\varphi \in \mathcal{C}(\mathcal{L})$. Then either $\varphi(v_n) = e_n$ or $\varphi(v_n) = -e_n$. If $\varphi(v_n) = e_n$ then $\varphi \in \mathcal{C}(\mathcal{L}')$. If $\varphi(v_n) = -e_n$ then $\tau\varphi \in \mathcal{C}(\mathcal{L}')$. Thus we have an isomorphism $\mathcal{C}(\mathcal{L}') \times G \to \mathcal{C}(\mathcal{L})$ given by $(\varphi, g) \mapsto g\varphi$. So part 1 is shown.

Let us now prove part 2. After translation and rotation, we may as well assume that $G = O(2)$. If $\beta\colon Z' \times SO(2) \to Z$ is the map $\beta(z, g) = gz$, let $\beta^{-1} = (\eta, \gamma)$ for entire rational functions $\eta\colon Z \to Z'$ and $\gamma\colon Z \to SO(2)$. Let

$$Z'' = \{(z, w) \in Z \times \mathbb{R}^n \mid w = \gamma(z) e_{n-1}\}$$

Note that $Z' \times e_{n-1} \subset Z''$ and is invariant under $O(1)$. So by part 1 we have a connected classical linkage $\mathcal{L}'$ with only n fixed vertices $v_0, \ldots, v_{n-1}$, fixed at 0 and e_i, and so that there is a finite F and an analytic function $\alpha'\colon Z' \times F \to (\mathbb{R}^n)^m$ so that

$$\mathcal{C}(\mathcal{L}') = \{\, (x, e_{n-1}, \alpha'(x, f)) \mid x \in Z' \text{ and } \ f \in F\,\}$$

and so that the map $(x, f) \mapsto (x, e_{n-1}, \alpha'(x, f))$ is an analytic isomorphism.

We obtain $\mathcal{L}$ by just unfixing the vertex v_{n-1}, but adding edges $\overline{v_{n-1} v_0}$ of length 1 and $\overline{v_{n-1} v_i}$ of length $\sqrt{2}$ for $1 \le i \le n-2$, if these are not already present. If $\varphi \in \mathcal{C}(\mathcal{L})$, then $\varphi(v_{n-1})$ lies on the circle of radius 1 about 0 in the $x_{n-1}x_n$ plane. So there is a unique $g \in SO(2)$ so that $g^{-1}\varphi \in \mathcal{C}(\mathcal{L})$, and g is a polynomial function of $\varphi(v)$. So we get an isomorphism $\beta'\colon \mathcal{C}(\mathcal{L}') \times SO(2) \to \mathcal{C}(\mathcal{L})$ given by $\beta'(\varphi, g) = g\varphi$. Define $\alpha\colon Z \times F \to (\mathbb{R}^n)^m$ by $\alpha(x, f) = (\gamma(x) e_{n-1}, \gamma(x)\alpha'(\gamma(x)^{-1}x, f))$. Note that the map $(x, f) \mapsto (x, \alpha(x, f)$ is a composition of the analytic isomorphisms $\beta^{-1} \times id\colon Z \times F \to Z' \times SO(2) \times F$ and $(z, g, f) \mapsto ((z, e_{n-1}, \alpha'(z, f)), g) \in \mathcal{C}(\mathcal{L}') \times SO(2)$ and β'. Thus it is an analytic isomorphism from $Z \times F$ to $\mathcal{C}(\mathcal{L})$.

So it only remains to prove the indicated converse of part 2. So suppose $\mathcal{L}$ is a connected planar classical linkage with one fixed vertex v_1 fixed at z_1, and some other nonfixed vertices $v_2, \ldots, v_k$. After reordering the vertices, we may suppose that $\overline{v_1 v_k}$ is an edge of $\mathcal{L}$. Let this edge have length r. Let $\mathcal{L}'$ be obtained from $\mathcal{L}$ by fixing v_k at some point z_k with $|z_k - z_1| = r$. Let $Z' = \mathcal{C}(\mathcal{L}')$. Let $G^+ \subset \mathrm{Euc}(2)^+$ be the group of rotations about z_1. Then $Z' \times G^+ \to \mathcal{C}(\mathcal{L})$ is an isomorphism since $\mathcal{C}(\mathcal{L})$ is G^+ invariant by Lemma 3.3, and for any $\varphi \in \mathcal{C}(\mathcal{L})$ there is a unique $g \in G^+$ which rotates $\varphi(v_k)$ to z_k, and this g is a polynomial function of $\varphi(v_k)$. $\square$

Proof. (of Theorem 1.7) Let $Z'' = \beta(Z' \times SO(2))$. Note that Z'' is $SO(2)$ invariant. Also, the map $\eta\colon Z'' \times \mathrm{Tran}(2) \to Z$ given by $\eta(z, \tau) = \tau z$ is an isomorphism, since any $g \in \mathrm{Euc}(2)^+$ can be uniquely decomposed as $g = \tau\gamma$ for $\tau \in \mathrm{Tran}(2)$ and $\gamma \in SO(2)$.

By Theorem 1.6 there is a connected classical linkage $\mathcal{L}'$ with just one fixed vertex, a finite set F, and an analytic $\alpha' \colon Z'' \times F \to (\mathbb{R}^2)^m$ so that

$$\mathcal{C}(\mathcal{L}') = \{(x, \alpha'(x, f)) \mid x \in Z'' \text{ and } f \in F\} \tag{5.2}$$

and in fact the map $(x, f) \mapsto (x, \alpha'(x, f))$ is an analytic isomorphism from $Z'' \times F$ to $\mathcal{C}(\mathcal{L}')$. Moreover, the fixed vertex v is fixed at 0. Let $\mathcal{L}$ be formed from $\mathcal{L}'$ by unfixing the vertex v. Let $\alpha \colon Z \times F \to (\mathbb{R}^2)^m$ be the analytic map $\alpha(z, f) = \tau\alpha'(\tau^{-1}z, f)$ where $\tau \in \mathrm{Tran}(2)$ is the unique translation so that $\tau^{-1}z \in Z''$. To be precise, $\eta^{-1}(z) = (\tau^{-1}z, \tau)$.

Note that the map $(z, f) \mapsto (z, \alpha(z, f))$ is the composition of analytic isomorphisms $\eta^{-1} \times id \colon Z \times F \to Z'' \times \mathrm{Tran}(2) \times F$ and $(z, \tau, f) \mapsto ((z, \alpha'(z, f)), \tau) \in \mathcal{C}(\mathcal{L}') \times \mathrm{Tran}(2)$ and $(\varphi, \tau) \mapsto \tau\varphi \in \mathcal{C}(\mathcal{L})$ (which is an isomorphism by Lemma 3.4). Hence it gives an analytic isomorphism from $Z \times F$ to $\mathcal{C}(\mathcal{L})$ as desired.

So the first part of Theorem 1.7 is proven. Now suppose that $\mathcal{L}$ is a connected classical linkage with no fixed vertices and at least two vertices. We may then pick two vertices v and w of $\mathcal{L}$ so that $\overline{vw}$ is an edge of $\mathcal{L}$. Let $\mathcal{L}'$ be the linkage obtained from $\mathcal{L}$ by fixing v to 0 and fixing w to some point z_0 with $|z_0| = \ell(\overline{vw})$. Note that $\mathcal{C}(\mathcal{L}') \subset \mathcal{C}(\mathcal{L})$. Moreover the map $\mathcal{C}(\mathcal{L}') \times \mathrm{Euc}(2)^+ \to \mathcal{C}(\mathcal{L})$ is an analytic isomorphism since for any $\varphi \in \mathcal{C}(\mathcal{L})$ we know that $\gamma\varphi \in \mathcal{C}(\mathcal{L}')$ where $\gamma \in \mathrm{Euc}(2)^+$ is the unique element so $\gamma\varphi(v) = 0$ and $\gamma\varphi(w) = z_0$, and γ is a polynomial function of $\varphi(v)$ and $\varphi(w)$. $\square$

6. Which Functions have Functional Linkages?

Suppose $\mathcal{L}$ is a (quasi)functional linkage for some function f with domain X. What functions $f \colon X \to (\mathbb{R}^n)^m$ are possible? We can completely characterize quasifunctional linkages as follows:

Theorem 6.1. *Suppose $X \subset (\mathbb{R}^n)^k$ and $f \colon X \to (\mathbb{R}^n)^m$ is a map. Then the following are equivalent:*

1. *There is a quasifunctional linkage $\mathcal{L}$ for f with domain X.*
2. *There is a classical quasifunctional linkage $\mathcal{L}$ for f with domain X.*
3. *The graph of f is a semialgebraic set and after perhaps permuting the $\mathbb{R}^n$ factors we have:*
   - (a) *$X = Y_0 \times Y_1 \times \cdots Y_\ell$, where $Y_i \subset (\mathbb{R}^n)^{k_i}$, (and $k_0 = 0$ is allowed, but $k_i \geq 1$ for $i \geq 1$).*
   - (b) *f is a product of maps $f_i \colon Y_i \to (\mathbb{R}^n)^{m_i}$, (where $m_i = 0$ is allowed and corresponds to composition with projection).*
   - (c) *Y_0 is compact.*
   - (d) *If $i \geq 1$ then Y_i is invariant under the action of $\mathrm{Euc}(n)$ with compact quotient.*
   - (e) *If $i \geq 1$, f_i is $\mathrm{Euc}(n)$ equivariant. That is, for every $\beta \in \mathrm{Euc}(n)$ we have $\beta(f_i(z)) = f_i(\beta(z))$.*

Proof. It is trivial that 2 implies 1. Let us see why 1 implies 3. If q and p are the input and output maps of $\mathcal{L}$, then the graph of f is the image of the polynomial map $(q, p) \colon \mathcal{C}(\mathcal{L}) \to (\mathbb{R}^n)^k \times (\mathbb{R}^n)^m$. So the graph of f is semialgebraic by [6]. By Lemma 6.5 below, we may as well assume that $\mathcal{L}$ is connected. If $\mathcal{L}$ has any fixed vertices, then $\mathcal{C}(\mathcal{L})$ is compact by Theorem 1.4, so X is compact since it is the image of the input map q. So we may then take $\ell = 0$ and part 3 will hold true. On the other hand, if $\mathcal{L}$ has no fixed vertices then $\mathcal{C}(\mathcal{L})$ is invariant under the action of $\mathrm{Euc}(n)$ with compact quotient by Theorem 1.4. We take $\ell = 1$ and $Y_0 =$ a point. Note that the input and output maps q and p are $\mathrm{Euc}(n)$ equivariant. Consequently $Y_1 = q(\mathcal{C}(\mathcal{L}))$ is invariant under the action of $\mathrm{Euc}(n)$ with compact quotient (since q induces a continuous map from $\mathcal{C}(\mathcal{L})/\mathrm{Euc}(n)$ onto $Y_1/\mathrm{Euc}(n)$). Note that f is $\mathrm{Euc}(n)$ equivariant since if $\beta \in \mathrm{Euc}(n)$ we have

$$f(\beta z) = f(\beta q(y)) = f(q(\beta y)) = p(\beta y) = \beta p(y) = \beta f(q(y)) = \beta f(z)$$

for all $z \in Y_1$ and $y \in q^{-1}(z)$. So the implication 1 implies 3 is shown.

Now let us show that 3 implies 2. Note that the graph G_i of f_i is a semialgebraic set since it is a projection of the graph of f. If $i > 0$ then by 3d) and 3e), the graph G_i is invariant under the action of $\mathrm{Euc}(n)$ with compact quotient. By 3c), G_0 is compact. Consequently, by Theorem 1.5 there are classical linkages $\mathcal{L}_i$ and $U_i \subset \mathcal{V}(\mathcal{L}_i)$ so that $\mathcal{SC}(\mathcal{L}_i, U_i) = G_i$. Since Y_i is a coordinate projection of G_i, there is a $W_i \subset U_i$ so that $\mathcal{SC}(\mathcal{L}_i, W_i) = Y_i$. Letting W_i be the input vertices and $U_i - W_i$ be the output vertices, we thus get a quasifunctional linkage for f_i with domain Y_i. By 3a) and 3b) and Lemma 4.2 we then see that the disjoint union of the $\mathcal{L}_i$ is a quasifunctional linkage for f. □

Now that quasifunctional linkages are classified, we attempt to classify functional linkages. This is a bit trickier. For example, the function $x \mapsto |x_1| e_1$ with domain the cube $[-1, 1]^n$ has a quasifunctional linkage since its graph is compact and semialgebraic. But it has no functional linkage whose restricted domain includes a neighborhood of the point 0. This is because if $\mathcal{L}$ is a functional linkage for f with restricted domain U, then $f|_U$ must be an analytic function, since if q and p are the input and output maps, there is a finite set F and an analytic isomorphism $\sigma\colon U \times F \to q^{-1}(U)$ so that $q(\sigma(u, c)) = u$ for all $c \in F$. Since $\mathcal{L}$ is functional, we know that $p(\sigma(u, c)) = f(u)$ and thus f is the composition of two analytic functions.

The following two theorems are restricted to the compact case because we don't know fine enough information about noncompact configuration spaces.

Theorem 6.2. *Suppose $X \subset (\mathbb{R}^n)^k$ is compact and $f\colon X \to (\mathbb{R}^n)^m$ is a map, and $U \subset X$. Then the following are equivalent:*

1. *There is a functional classical linkage $\mathcal{L}$ for f with domain X and restricted domain U.*
2. *There are a compact real algebraic set Y, and polynomial maps $q\colon Y \to X$ and $p\colon Y \to (\mathbb{R}^n)^m$, a finite set F and an analytic map $\sigma\colon U \times F \to Y$ so that:*
   (a) *$p = f \circ q$.*
   (b) *q is onto.*
   (c) *$q\sigma(x, c) = x$ for all $(x, c) \in U \times F$.*
   (d) *σ is an analytic isomorphism onto $q^{-1}(U)$.*

Proof. The implication 1 implies 2 follows immediately from the definition of functional linkage and the fact that configuration spaces of classical linkages are algebraic sets.

Let us now see why 2 implies 1. So suppose we have Y, p, q, F, and σ as above. By replacing Y with the graph of (p, q) we may as well assume that p and q are given by coordinate projection. So $Y \subset (\mathbb{R}^n)^m \times (\mathbb{R}^n)^k \times (\mathbb{R}^n)^b$ and p and q are induced by projections to the first and second batches of coordinates. By Theorem 1.1, there is a classical linkage $\mathcal{L}$, a finite set G and an analytic $\beta\colon Y \times G \to (\mathbb{R}^n)^\ell$ so that

$$\mathcal{C}(\mathcal{L}) = \{(y, \beta(y, c)) \mid y \in Y \text{ and } c \in G\}$$

and so the map $(y, c) \mapsto (y, \beta(y, c))$ is an isomorphism from $Y \times G$ to $\mathcal{C}(\mathcal{L})$. Then $\mathcal{L}$ is functional for f with restricted domain U. The first m vertices are the output vertices, the next k are the input vertices, and the map $\gamma\colon U \times F \times G \to \mathcal{C}(\mathcal{L})$ giving the trivial analytic cover is $\gamma(x, c, d) = (\sigma(x, c), \beta(\sigma(x, c), d))$. □

Theorem 6.3. *Suppose $X \subset (\mathbb{R}^n)^k$ is compact and $f\colon X \to (\mathbb{R}^n)^m$ is a map. Then the following are equivalent:*

1. *There is a strong functional linkage $\mathcal{L}$ for f with domain X.*
2. *There is a quasialgebraic set Y, polynomial maps $q\colon Y \to X$ and $p\colon Y \to (\mathbb{R}^n)^m$ so that:*
   (a) *$p = f \circ q$.*
   (b) *q is an analytic isomorphism onto X.*

Proof. One direction follows immediately from the definition of strong functional linkage and the fact that configuration spaces of cabled linkages are quasialgebraic sets.

So suppose we have Y, p, and q as above. By replacing Y with the graph of (p, q) we may as well assume that p and q are given by coordinate projection. So $Y \subset (\mathbb{R}^n)^m \times (\mathbb{R}^n)^k \times (\mathbb{R}^n)^b$ and p and q are induced by projections to the first and second batches of coordinates. By Theorem 1.2, there is a cabled linkage $\mathcal{L}$ and an analytic $\beta\colon Y \to (\mathbb{R}^n)^\ell$ so that

$$\mathcal{C}(\mathcal{L}) = \{(y, \beta(y)) \mid y \in Y\}$$

Then $\mathcal{L}$ is strongly functional for f with domain X. The first m vertices are the output vertices, the next k are the input vertices. The input map is projection to Y, followed by q and is hence an analytic isomorphism. □

Theorems 6.2 and 6.3 were restricted to the compact case and so did not completely classify functional linkages. In the planar case, however, we have enough information to completely classify functional classical linkages.

Theorem 6.4. *Suppose $X \subset (\mathbb{R}^2)^k$ and $f\colon X \to (\mathbb{R}^2)^m$ is a map, and $U \subset X$ is nonempty and open. Then the following are equivalent:*

1. *There is a functional classical linkage $\mathcal{L}$ for f with domain X and restricted domain U.*
2. *After perhaps permuting the $\mathbb{R}^2$ factors we have:*
   (a) *$X = X_0 \times X_1 \times \cdots \times X_\ell$, where $X_i \subset (\mathbb{R}^2)^{k_i}$, (and $k_0 = 0$ is allowed, but $k_i \geq 1$ for $i \geq 1$).*
   (b) *f is a product of maps $f_i\colon X_i \to (\mathbb{R}^2)^{m_i}$, (where $m_i = 0$ is allowed and corresponds to composition with projection).*
   (c) *For $i = 0, \ldots, \ell$ there are real algebraic sets Y_i and polynomial maps $q_i\colon Y_i \to X_i$ and $p_i\colon Y_i \to (\mathbb{R}^2)^{m_i}$ and a finite set F and an analytic isomorphism $\sigma\colon U \times F \to (q_0 \times \cdots \times q_\ell)^{-1}(U)$ so that:*
      (i) *$p_i = f_i \circ q_i$*
      (ii) *q_i is onto.*

(iii) $(q_0 \times \cdots \times q_\ell)(\sigma(x,c)) = x$ *for all* $(x,c) \in U \times F$.

(d) Y_0 *is compact.*

(e) *If* $i \geq 1$ *then* X_i *and* Y_i *are invariant under the action of* Euc(2) *with compact quotient.*

(f) *If* $i \geq 1$ *then* p_i *and* q_i *are* Euc(2) *equivariant.*

(g) *If* $i \geq 1$ *and* $m_i > 0$ *then there is a compact real algebraic subset* $Y_i' \subset Y_i$ *so that the map* $(y, g) \mapsto gy$ *is an isomorphism from* $Y_i' \times \mathrm{Euc}(2)^+$ *to* Y_i.

Proof. First let us prove that 2 implies 1. By replacing each Y_i by the graph of (q_i, p_i), we may as well assume that p_i and q_i are given by coordinate projections. By Theorems 1.6 and 1.7, there are connected classical linkages $\mathcal{L}_i$, finite sets F_i, and analytic $\alpha_i \colon Y_i \times F_i \to (\mathbb{R}^2)^{\ell_i}$ so that

$$\mathcal{C}(\mathcal{L}_i) = \{(x, \alpha_i(x,c)) \mid x \in Y_i \text{ and } c \in F_i\} \tag{6.1}$$

and the map $(x,c) \mapsto (x, \alpha_i(x,c))$ is an analytic isomorphism. Moreover, if $i \geq 1$, $\mathcal{L}_i$ has no fixed vertex. Let $\mathcal{L}$ be the disjoint union of the $\mathcal{L}_i$. Recall by Lemma 3.2 that $\mathcal{C}(\mathcal{L}) = \mathcal{C}(\mathcal{L}_0) \times \cdots \times \mathcal{C}(\mathcal{L}_\ell)$.

The projection maps $\pi_i \colon \mathcal{C}(\mathcal{L}_i) \to Y_i$ are trivial analytic coverings. Hence if $\pi = \pi_0 \times \cdots \times \pi_\ell$, then $\pi \colon \mathcal{C}(\mathcal{L}) \to Y_0 \times \cdots \times Y_\ell$ is a trivial analytic covering. So if $q \colon \mathcal{C}(\mathcal{L}) \to X$ is defined by $q = (q_0 \times \cdots \times q_\ell) \circ \pi$, we know by c) that q restricts to a trivial analytic covering $q| \colon q^{-1}(U) \to U$.

Let $p = (p_0 \times \cdots \times p_\ell) \circ \pi$. Since p_i and q_i are given by coordinate projections from Y_i, we know $p_i \circ \pi$ and $q_i \circ \pi$ are given by coordinate projections from $\mathcal{C}(\mathcal{L})$. Consequently, there are vertices $w_1, \ldots, w_k$ and $v_1, \ldots, v_m$ of $\mathcal{L}$ so that

$$\begin{aligned} q &= (\rho_{\mathcal{L},w_1}, \ldots, \rho_{\mathcal{L},w_k}) \\ p &= (\rho_{\mathcal{L},v_1}, \ldots, \rho_{\mathcal{L},v_m}) \end{aligned}$$

Let the w_i be input vertices and the v_i be output vertices. Then the input and output maps are q and p respectively. Note that

$$p = p_0 \circ \pi_0 \times \cdots \times p_\ell \circ \pi_\ell = f_0 \circ q_0 \circ \pi_0 \times \cdots \times f_\ell \circ q_\ell \circ \pi_\ell = f \circ q$$

so $\mathcal{L}$ is quasifunctional for f. Also the domain of $\mathcal{L}$ is $q(\mathcal{C}(\mathcal{L})) = q_0(Y_0) \times \cdots \times q_\ell(Y_\ell) = X$ by 2(c)(ii). Consequently $\mathcal{L}$ is a functional linkage with domain X and restricted domain U. So the implication 2 implies 1 is shown.

Now let us show that 1 implies 2. Let $\mathcal{L}$ be a functional classical linkage for f with domain X and restricted domain U. We may write $\mathcal{L}$ as the disjoint union of $\mathcal{L}_i$ for $i = 0, \ldots, \ell$ so that $\mathcal{L}_0$ is the union of components of $\mathcal{L}$ which have fixed vertices and so each $\mathcal{L}_i$ for $i \geq 1$ is connected and has no fixed vertices. Let $Y_i = \mathcal{C}(\mathcal{L}_i)$.

By Lemma 6.5 below, we know that each $\mathcal{L}_i$ is quasifunctional for some $f_i \colon X_i \to (\mathbb{R}^2)^{m_i}$ and that 2a) and b) hold. Let q and p be the input and output maps for f. Note that $q = q_0 \times \cdots \times q_\ell$ and $p = p_0 \times \cdots \times p_\ell$. Then $p = f \circ q$ implies that $p_i = f_i \circ q_i$ for each i. Also each q_i maps onto X_i since q maps onto X. Note that 2(c)(iii) holds by definition of restricted domain. By Theorem 1.4 we know X_0 is compact since it is the projection of the compact $\mathcal{C}(\mathcal{L}_0)$. By part 1 of Theorem 1.4 we know 2e), and 2f) follows as in the proof of Theorem 6.1. Finally 2g) follows from Theorem 1.7. □

We used above the following converse to Lemma 4.2.

Lemma 6.5. *Suppose* $\mathcal{L}$ *is a quasifunctional linkage for* $f \colon (\mathbb{R}^n)^k \to (\mathbb{R}^n)^m$ *and* $\mathcal{L}$ *is the disjoint union of two linkages* $\mathcal{L}_0$ *and* $\mathcal{L}_1$. *Then each* $\mathcal{L}_i$ *is a quasifunctional* *linkage for some* $f_i\colon (\mathbb{R}^n)^{k_i} \to (\mathbb{R}^n)^{m_i}$ *and after perhaps reordering the coordinates,* $f = f_0 \times f_1$. *The input and output maps for* f *are also the products of the input and output maps for the* f_i.

Moreover, if $\mathcal{L}$ *is a (strong) functional linkage with restricted domain* U *and* U *is a nonempty open subset of the domain of* f*, then the* $\mathcal{L}_i$ *will also be (strong) functional linkages with restricted domain* U_i*, with* U_i *nonempty and open in the domain of* f_i.

Proof. Let $w_1, \dots, w_k$ and $v_1, \dots, v_m$ be the input and output vertices of $\mathcal{L}$. Reorder these so that $w_i \in \mathcal{L}_0$ if and only if $i \le k_0$ and $v_i \in \mathcal{L}_0$ if and only if $i \le m_0$. Let $k_1 = k - k_0$ and $m_1 = m - m_0$. Recall that $\mathcal{C}(\mathcal{L}) = \mathcal{C}(\mathcal{L}_0) \times \mathcal{C}(\mathcal{L}_1)$ by Lemma 3.2. Let $q\colon \mathcal{C}(\mathcal{L}_0) \times \mathcal{C}(\mathcal{L}_1) \to (\mathbb{R}^n)^{k_0} \times (\mathbb{R}^n)^{k_1}$ be the input map. Note that $q = q_0 \times q_1$ for some $q_i\colon \mathcal{C}(\mathcal{L}_i) \to (\mathbb{R}^n)^{k_i}$ and likewise the output map p of $\mathcal{L}$ is $p_0 \times p_1$. Choose any $\varphi_i' \in \mathcal{C}(\mathcal{L}_i)$ and let $q(\varphi_0', \varphi_1') = (x_0', x_1')$. If $\mathcal{L}$ is a (strong) functional linkage with nonempty open restricted domain U, choose them so that $(x_0', x_1') \in U$.

We may write f as $f(x_0, x_1) = (f_0'(x_0, x_1), f_1'(x_0, x_1))$ where $x_i \in (\mathbb{R}^n)^{k_i}$ and $f_i'(x_0, x_1) \in (\mathbb{R}^n)^{m_i}$. Since $f \circ (q_0 \times q_1) = p_0 \times p_1$ we know that $f_i' \circ (q_0 \times q_1) = p_i$. So for any $\varphi_i \in \mathcal{C}(\mathcal{L}_i)$,

$$f_0'(q_0(\varphi_0), q_1(\varphi_1)) = p_0(\varphi_0) = f_0'(q_0(\varphi_0), x_1')$$

and thus $f_0'(x_0, x_1)$ is independent of the x_1 coordinate and we may define $f_0(x_0) = f_0'(x_0, x_1')$. Similarly, f_1' is independent of x_0 and we may define $f_1(x_1) = f_1'(x_0', x_1)$. Then $f_i \circ q_i = p_i$ and $\mathcal{L}_i$ is a quasifunctional linkage for f_i. Also $f = f_0 \times f_1$.

Note that the domain of f is the cartesian product of the domains of f_i. So if $\mathcal{L}$ is a (strong) functional linkage with open restricted domain U, then after restricting the domain further, we may as well suppose that $U = U_0 \times U_1$ for $U_i \subset (\mathbb{R}^n)^{k_i}$, and $x_i' \in U_i$ and U_i open in the domain of f_i. In the strong case, we also may as well assume that U and U_i are the domains of f and f_i. In the nonstrong case, we may restrict each U_i further and assume it is connected. We may do this because the domain of f_i is a semialgebraic set, hence it is locally connected by, for example, the curve selection lemma [5] or triangulability [4].

By (strong) functionality, there is a finite set F and an analytic isomorphism

$$\sigma\colon U_0 \times U_1 \times F \to q^{-1}(U_0 \times U_1) = q_0^{-1}(U_0) \times q_1^{-1}(U_1)$$

so that $q\sigma(x_0, x_1, c) = (x_0, x_1)$ and so that F is a single point in the strong case.

Let $F_0 \subset F$ be the subset so that $x_0' \times x_1' \times F_0 = \sigma^{-1}(q_0^{-1}(x_0') \times \varphi_1')$. By connectedness of U_0 and discreteness of F (in the nonstrong case), or by the fact that $F_0 = F =$ a point (in the strong case), we then know that $x_0 \times x_1' \times F_0 = \sigma^{-1}(q_0^{-1}(x_0) \times \varphi_1')$ for any $x_0 \in U_0$. We thus get an analytic isomorphism $\sigma_0\colon U_0 \times F_0 \to q_0^{-1}(U_0)$ given by $(\sigma_0(u, c), \varphi_1') = \sigma(u, x_1', c)$. So $\mathcal{L}_0$ is a (strong) functional linkage for f_0 with restricted domain U_0. Similarly, f_1 is a (strong) functional linkage with restricted domain U_1. $\square$

Remark 4. Note that in Lemma 6.5 we have $U_0 \times U_1 \subset U$ and in fact, we may choose U_i so that $U_0 \times U_1$ contains any one given point of U.

7. Simulating Higher Dimensional Linkages

Suppose K is a finite simplicial complex and we identify each simplex of K with a particular (linear) simplex in some euclidean space. We could then study the configuration space of all realizations of K in some $\mathbb{R}^n$, i.e., all maps of K to

$\mathbb{R}^n$ which restrict to an isometry on each simplex of K. However, by Lemma 7.3 and Lemma 7.4 below, a simplex in $\mathbb{R}^n$ is determined up to Euclidean motions by the lengths of its edges. Thus the configuration space of K is the same as the configuration space of its one dimensional skeleton, the union of its vertices and edges. Consequently by looking at higher dimensional simplicial linkages, we get nothing different in the way of configuration spaces.

We could generalize even further and still get no new configuration spaces. We could consider configuration spaces of polyhedra glued together, but we could still simulate such objects by a one dimensional linkage. In particular, let K_i, $i = 1, \ldots m$ be realizations of finite simplicial complexes in some Euclidean space. Let $K_{ji} \subset K_i$ be possibly empty subcomplexes, $j = 1, \ldots, i-1, i+1, \ldots, m$, and let $\varphi_{ji} \colon K_{ji} \to K_{ij}$ be compatible simplicial isometries. In particular, $\varphi_{ji} = \varphi_{ij}^{-1}$, $\varphi_{ji}(K_{ji} \cap K_{ki}) = K_{ij} \cap K_{kj}$ and $\varphi_{kj}(\varphi_{ji}(x)) = \varphi_{ki}(x)$ for all $x \in K_{ij} \cap K_{kj}$. Since φ_{ji} is simplicial, it takes vertices to vertices, edges to edges, etc. We may form an object K by gluing the K_i together using the maps φ_{ji}. We may then look at the configuration space of maps from K to $\mathbb{R}^n$ which restrict to an isometry on each K_i. For example K could be the surface of a unit cube and the K_i are its faces.

But the configuration space of K is the same as the configuration space of some one dimensional linkage K'. To construct K', we just replace each K_i with the one dimensional complex K_i' which has the same vertices as K_i and has one edge of the appropriate length for each pair of vertices of K_i. Glue these one dimensional complexes together using the maps φ_{ji} to identify vertices. Also identify edges between pairs of identified vertices, and thus obtain a one dimensional simplicial complex K'. In our unit cube example we would get a one dimensional complex consisting of the edges of the unit cube and the two diagonals on each of its faces. By Lemma 7.4 below, the isometries of K_i and K_i' are the same, so the resulting configuration spaces of K and K' are the same.

Of course if we generalize further to higher dimensional linkages with curved faces, such reductions are no longer possible.

Following are Lemmas proving various results referred to above. No doubt these are well known.

Lemma 7.1. *Suppose $z_i \in \mathbb{R}^n$, $i = 1, \ldots k$. Let T be the affine span of $z_1, \ldots, z_k$. Then for any fixed $z \in T$ the distance from z to any point $x \in \mathbb{R}^n$ is determined by the distances from the z_i to x.*

Proof. We may write $z = \sum_{i=1}^k t_i z_i$ where $\sum_{i=1}^k t_i = 1$. Then

$$\begin{aligned} |x - z|^2 &= |\sum_{i=1}^k t_i (x - z_i)|^2 \\ &= \sum_{i,j} t_i t_j (x - z_i) \cdot (x - z_j) \end{aligned}$$

But $(x - z_i) \cdot (x - z_j) = (|x - z_i|^2 + |x - z_j|^2 - |z_i - z_j|^2)/2$. □

Lemma 7.2. *Suppose $K \subset \mathbb{R}^m$ and suppose $\alpha \colon K \to \mathbb{R}^n$ is an affine transformation which restricts to an isometry on K. Then α restricts to an isometry on the affine span of K.*

Proof. If S is the affine span of K and $x, y \in S$ then there are $x_i \in K$, $i = 1, \ldots, k$ and t_i and s_i so that $x = \sum_{i=1}^k t_i x_i$, $y = \sum_{i=1}^k s_i x_i$, and $\sum_{i=1}^k t_i = \sum_{i=1}^k s_i = 1$.

Since α restricts to an isometry on K, we know that $(\alpha x_i - \alpha x_\ell) \cdot (\alpha x_j - \alpha x_\ell) = (x_i - x_\ell) \cdot (x_j - x_\ell)$ for all i, j, ℓ. So

$$\begin{aligned} |\alpha x - \alpha y|^2 &= |\sum_{i=1}^{k} (t_i - s_i)(\alpha x_i - \alpha x_1)|^2 \\ &= \sum_{i,j} (t_i - s_i)(t_j - s_j)(\alpha x_i - \alpha x_1) \cdot (\alpha x_j - \alpha x_1) \\ &= |\sum (t_i - s_i)(x_i - x_1)|^2 = |x - y|^2 \end{aligned}$$

□

Lemma 7.3. *Let $K \subset \mathbb{R}^m$ be the realization of some compact simplicial complex and suppose $\varphi_i \colon K \to \mathbb{R}^n$, $i = 0, 1$ are two isometric embeddings. Then there is a $\beta \in \mathrm{Euc}(n)$ so that $\varphi_1 = \beta \circ \varphi_0$. There is also an affine transformation $\alpha \colon \mathbb{R}^m \to \mathbb{R}^n$ so that φ_0 is the restriction of α.*

Proof. Suppose first that K is zero dimensional, so K is a finite number of points $x_i \in \mathbb{R}^m$, $i = 1, \dots, k$. By induction on k, after composing φ_1 with some $\beta' \in \mathrm{Euc}(n)$ we may as well assume that there is an affine transformation $\alpha' \colon \mathbb{R}^m \to \mathbb{R}^n$ so that $\alpha'(x_i) = \varphi_0(x_i) = \varphi_1(x_i)$ for all $i \neq k$. Let T be the affine span of $\varphi_0(x_1), \dots, \varphi_0(x_{k-1})$. By Lemma 7.1 we know that $\varphi_0(x_k)$ and $\varphi_1(x_k)$ have the same distance to each point of T. In particular, their orthogonal projections to T coincide since the orthogonal projection is the point on T of minimum distance. So we may find a $\beta \in \mathrm{Euc}(n)$ which fixes T and takes $\varphi_0(x_k)$ to $\varphi_1(x_k)$. (If z_0 is their common orthogonal projection, choose a rotation $Q \in O(n)$ so $Q(\varphi_0(x_k) - z_0) = \varphi_1(x_k) - z_0$ and so Q fixes $(\varphi_0(x_k) - z_0)^\perp \cap (\varphi_1(x_k) - z_0)^\perp$. Let $\beta(z) = z_0 + Q(z - z_0)$.)

Let us now construct α. Since α' restricts to an isometry on $x_1, \dots x_{k-1}$, it restricts to an isometry of the affine span S of $x_1, \dots x_{k-1}$ by Lemma 7.2. So if $x_k \in S$ then α' restricts to an isometry of K and hence $\alpha'(x_k) = \varphi_0(x_k)$ by Lemma 7.1. If $x_k \notin S$, we may by hand pick an affine α so α restricts to α' on S but $\alpha(x_k) = \varphi_0(x_k)$. In particular, if $\alpha(t(x_k - x) + y) = t(\varphi_0(x_k) - \alpha'(x)) + \alpha'(y)$ for all $x, y \in S$ and $t \in \mathbb{R}$, this defines α on the affine span of $x_1, \dots, x_k$ and you just extend α in any fashion to an affine map on all of $\mathbb{R}^n$.

Now suppose K has dimension $k > 0$. By induction on k, we may as well assume that there is an affine transformation $\alpha \colon \mathbb{R}^m \to \mathbb{R}^n$ so that $\alpha(x) = \varphi_0(x) = \varphi_1(x)$ for all x in the $k-1$ skeleton of K. Take any point x in the interior of a k simplex σ of K, and let $v_0, \dots, v_k$ be the vertices of σ, so $x = \sum_{i=0}^{k} t_i v_i$ and $\sum_{i=0}^{k} t_i = 1$. Let $x' = (x - t_k v_k)/(1 - t_k)$, so x is on the line segment from x' to v_k. Then since φ_i is an isometry we know that

$$\varphi_i(x) = t_k \varphi_i(v_k) + (1 - t_k)\varphi_i(x') = t_k \alpha(v_k) + (1 - t_k)\alpha(x') = \alpha(x)$$

□

Lemma 7.4. *Let $K \subset \mathbb{R}^m$ be the realization of some compact simplicial complex and let $K' \subset K$ be the set of vertices of K. Then any isometry $\varphi \colon K' \to \mathbb{R}^n$ is the restriction of a unique isometry $\varphi' : K \to \mathbb{R}^n$.*

Proof. By Lemma 7.3 there is an affine $\alpha \colon \mathbb{R}^m \to \mathbb{R}^n$ which restricts to φ on K'. But K is contained in the affine span of K', so by Lemma 7.2, α restricts to an isometry of K. Uniqueness follows from Lemma 7.3. □

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