Using Matlab for Numerical Evaluation of Multiple Integrals

Very few functions have integrals which can be written down as a formula involving well-known functions. Yet integration is important, so what do you do if you need to know the integral of something but you can’t find an antiderivative? You must turn to numerical techniques. A numerical technique will provide you with a number which usually is a good approximation of the integral. Note, before doing the examples below type in `syms x y z u v` to tell matlab these are your symbolic variables.

Single Integrals

You have already used the matlab function `quad` which allows you to perform a single integration. For example, to evaluate \( \int_2^4 \sin(x^3 - 7x) \, dx \) you could type:

```matlab
>> f = inline(vectorize(sin(x^3 - 7*x)),'x')
>> quad(f,2,4)
```

Matlab has a second numerical integrator called `quadl` which uses a different method from `quad`. As a rough check on accuracy you could integrate using `quadl` also by typing `quadl(f,2,4)` and compare the answers to see if they agree closely enough.

There are ways to specify how accurate you would like `quad` or `quadl` to be. Type `help quad` to find out how. Generally, you need not worry about this and just use default values.

Double Integrals

Matlab has a command `dblquad` which allows you to automatically evaluate a double integral over a rectangular region. For example, to evaluate \( \int_3^4 \int_1^3 \sqrt{x^2 + 4/y^2} \, dx \, dy \) type:

```matlab
>> f = inline(vectorize(sqrt(x^2 + 4/y^2)),'x','y')
>> dblquad(f,3,4,1,2)
```

Note that the limits of the \( x \) variable come first, then the limits of the \( y \) variable since this is the order the variables were specified in the inline function \( f \).

If the region \( R \) you are integrating over is not rectangular, but is vertically or horizontally simple, then you have a number of options:

1) You could do a change of variables to transform \( R \) into a square region. Then use `dblquad`.
2) You might be able to do the inner integration by hand if you are lucky. Then you have reduced the problem to a single integral which you can evaluate using `quad`.
3) You can use an mfile `ezint241.m`, available on my website, which will integrate for you.

A sample evaluation of a double integral

Let us consider the case where \( R \) is the triangular region \( \{(x,y) \mid x \geq 0, y \geq 0, x+y \leq 1\} \) and use each of the above methods to find \( \int_0^1 \int_0^{1-y} \sin(xy) \, dx \, dy \).

For the first method, use the change of variables \( y = u \) and \( x = v(1-u) \). Then the limits become \( 0 \leq u \leq 1 \) and \( 0 \leq v \leq 1 \). The Jacobian is \( \partial(x,y)/\partial(u,v) = u-1 \) which is negative on the region so

\[
\int_0^1 \int_0^{1-y} \sin(xy) \, dx \, dy = \int_0^1 \int_0^1 (1-u) \sin(uv(1-u)) \, du \, dv
\]

We type in:

```matlab
>> f = inline(vectorize(((1-u)*sin(u*v*(1-u))),'u','v'))
>> dblquad(f,0,1,0,1)
```

There is a standard way to do a change of variables which always changes an iterated integral to one with constant limits. If you have \( \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) \, dy \, dx \) then do the change of coordinates \( x = u, y = (1-v)g_1(u) + vg_2(u) \). Then the new limits are \( a \leq u \leq b, 0 \leq v \leq 1 \). Also the Jacobian is \( \partial(x,y)/\partial(u,v) = g_2(u) - g_1(u) \). So

\[
\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) \, dy \, dx = \int_a^b \int_0^1 f(u, (1-v)g_1(u) + vg_2(u)) \, (g_2(u) - g_1(u)) \, dv \, du
\]
assuming \( g_2(u) \geq g_1(u) \) for \( a \leq u \leq b \). For \( dx \, dy \) order just switch \( x \) and \( y \) as in the example above.

Now let us look at the second method. Note that
\[
\int_0^{1-y} \sin(xy) \, dx = -\frac{(\cos(xy))}{y} \bigg|_0^{1-y} = \frac{(1 - \cos(y(1-y)))}{y}
\]

So
\[
\int_0^1 \int_0^{1-y} \sin(xy) \, dx \, dy = \int_0^1 \frac{(1 - \cos(y(1-y)))}{y} \, dy
\]

So you could type in:
\[
>> f = \text{inline}((1-\text{cos}(y*(1-y)))/y)\,'y')
\]
\[
>> \text{quad}(f,0,1)
\]

Matlab might complain a bit when it evaluates \( f \) at 0 since it is dividing 0/0 although in fact the limit of this function at 0 is 0. If you can do it, this method is faster and more accurate than the others.

The third method relies on using the mfiles \texttt{ezint241.m} and \texttt{nquad.m}. Go to the course web site at http://www.math.umd.edu/~hck/241syl.html and you will be able to download these two files. Put these files in your home directory or in the directory or floppy you use to run Matlab. If you are using an owl lab PC and keep your work on a floppy, give Matlab the command \texttt{cd A:} to tell matlab you are using the floppy.

Now you are able to use the function \texttt{ezint241} to integrate. Type in:
\[
>> \text{ezint241}((\sin(x*y))/2, y,0,3-1.5*x, x,0,2)
\]

Here is what is going on. The first entry \( \sin(x*y) \) is the integrand. The next three entries are the innermost variable, \( x \), and its limits. The final three entries are the outer variable, \( y \), and its limits.

**Triple Integrals**

There are fewer options open to you if you need to calculate a triple integral. Matlab has no built-in triple integrator, so your only real options are the triple integral versions of methods 2 and 3 above.

So the best method is to see first if you can evaluate the inner integral, perhaps after changing the order of integration. If you can, you have reduced to a double integral and can use the methods of the previous section. If you can’t then you can use \texttt{ezint241} to evaluate your integral.

As an example, let us evaluate the integral
\[
\int_0^2 \int_0^{(6-3x)/2} \int_0^{6-3x-2y} \sin(x^2 e^y + z) \, dz \, dy \, dx
\]

In this case we can do the inner integration, so it is \( \int_0^2 \int_0^{(6-3x)/2} \left(\cos(x^2 e^y) - \cos(x^2 e^y + 6-3x-2y)\right) \, dy \, dx \). We now use the methods of the previous section and end up doing:
\[
>> \text{ezint241}((\cos(x^2*\text{exp}(y)))-\cos(x^2*\text{exp}(y)+6-3*x-2*y), y,0, 3-1.5*x, x,0,2)
\]

But suppose we didn’t see how to do the inner integration, or were just lazy and were willing to let the computer spend its time working out the triple integral. Then we could type:
\[
>> \text{ezint241}((\sin(x^2*\text{exp}(y))+z), z,0,6-3*x-2*y, y,0,3-1.5*x, x,0,2)
\]

**Computer assignment**

This assignment is due Thursday, Nov. 21 (or earlier, if you wish). You must provide me with printed copies of your output with your answers clearly indicated. Also show your work, for example the inner integral calculation you use for question 1. As usual, I strongly encourage you to work in groups of two or three, but no more. See the course web page for late breaking hints, tips, and tricks. In particular, \texttt{ezint241} was just written and you may need to be download a revised version.

1) Calculate \( \int_0^1 \int_0^2 \sin(y - x^2) \, dy \, dx \) in three ways, methods 1, 2 and 3. Are your answers comparable?

2) Find the area of the surface parameterized by \( F(u, v) = [uv, u^2 v, u] \) for \( 0 \leq u \leq 2, -1 \leq v \leq 1 \).

3) Calculate \( \int_0^1 \int_{1-x}^{15-y} \log(3 + x^2 + yz^2) \, dz \, dy \, dx \).