The Quotient vector Space

Suppose $V$ is a vector space over $K$ and $U \subset V$ is a subspace. We will describe a construction of the quotient vector space $V/U$.

But first we will discuss equivalence relations. If $S$ is a set then a relation $\sim$ on $S$ is some way of relating elements of $S$. The expression $x \sim y$ means $x$ is related to $y$. For example if $S$ is all the people in the world, then $x \sim y$ might mean ‘$x$ is a brother or sister of $y’$ and $x \sim y$ might mean ‘$x$ is a mother of $y’$. Both $\sim$ and $\approx$ are relations.

We say a relation $\sim$ is an equivalence relation if

a) $x \sim x$ for all $x \in S$.
b) If $x \sim y$ then $y \sim x$.
c) If $x \sim y$ and $y \sim z$ then $x \sim z$.

Thus in the examples above $\sim$ is an equivalence relation but $\approx$ is not. For a mathematical example, let $S$ be the integers and say that $x \sim y$ if and only if $x - y$ is even. This is an equivalence relation.

If you have an equivalence relation $\sim$ then $S$ can be divided up into equivalence classes. An equivalence class is a set of the form $[x] = \{y \in S \mid y \sim x\}$. Note that if $x \sim y$ then $[x] = [y]$. So two different equivalence classes are always disjoint. In the integer example above there are just two equivalence classes, the even integers and the odd integers. We let $S/\sim$ denote the set of equivalence classes of $S$. We say that $x$ is a representative of the equivalence class $[x]$. If $y \sim x$ then $y$ is also a representative of $[x]$.

Now let us go back to vector spaces. Suppose $V$ is a vector space over $K$ and $U \subset V$ is a subspace. We define an equivalence relation $\sim$ on $V$ by $x \sim y$ iff $x - y \in U$. Let $V/U = V/\sim$. Define addition and scalar multiplication on $V/U$ by

a) $[x] + [y] = [x + y]$
b) $c[x] = [cx]$

We must show these operation do not depend on which representative $x$ we choose. So suppose $[x] = [x']$ and $[y] = [y']$. We want to show that $[x + y] = [x' + y']$, i.e., that $x + y \sim x' + y'$. But $(x + y) - (x' + y') = x - x' + y - y'$ and $x - x' \in U$ and $y - y' \in U$. So $(x + y) - (x' + y') \in U$ which means $x + y \sim x' + y'$ which means $[x + y] = [x' + y']$. Likewise $(cx) - (c'x') = c(x - x') \in U$ so $[cx] = [c'x']$. You can show all the vector space axioms are satisfied, so $V/U$ is a vector space over $K$.

For example, let $V = \mathbb{C}^2$ and $U = \{(x, y) \mid x = 2y\}$. Then $(x, y) \sim (x', y')$ iff $x - x' = 2(y - y')$ iff $x - 2y = x' - 2y'$. Note that if $W$ is a complementry subspace to $U$ (i.e., $U \oplus W = V$) then for each $[(x, y)] \in U/V$ there is a unique representative $(x', y')$ in $W$. So we may identify $V/U$ with $W$. For example suppose $W$ is the $x$ axis. Then $(x, y) \sim (x - 2y, 0)$ on the $x$ axis.

For another example, let $V = \mathbb{R}^R$ and let $U = \{f \in V \mid f(t) = 0 \text{ for all } t \in [0, 1]\}$. Then two functions are equivalent if they agree on $[0, 1]$. Thus $V/U$ can be identified with $\mathbb{R}^{(0, 1)}$, the functions from $[0, 1]$ to $\mathbb{R}$. 